

Optimal Liquidation under Stochastic Resilience of Price Impact

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We solve explicitly a two-dimensional singular control problem of finite fuel type for infinite time horizon. The problem stems from the optimal liquidation of an asset position in a financial market with multiplicative price impact and stochastic resilience. The optimal control is obtained as a diffusion process reflected at a non-constant free boundary. To solve the HJB variational inequality and prove optimality, we show new results of independent interest on constructive approximations and Laplace transforms of the inverse local times for diffusions reflected at elastic boundaries.

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1 Introduction

Trading actions of large investors typically have an adverse effect on prices in financial markets. If liquidity is finite, trading large quantities over short time periods can cause high liquidity costs. Therefore, a large trader needs to balance her preference to complete a trading objective early against the wish to reduce liquidity costs. Considering the intertemporal modeling of price impact one can differentiate between permanent impact and temporary impact. Permanent impact of a trade is a price change which is of the same size for the current price and for all future prices. In contrast, temporary impact could be persistent but it lessens and eventually vanishes over time. Basically

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two approaches to temporary impact can be distinguished in the extensive literature on illiquid markets and, more specifically, on the optimal trade execution problem, see e.g. [GS13, LS13] and the references therein. The first approach takes temporary impact to be strictly instantaneous, like in the model by Almgren and Chriss, such that it does not depend on past trades but only on the present trade. Such an impact can be perceived as a (non-proportional, convex) transaction cost. A second approach, to which our paper belongs, is inspired by the idea of a shadow limit order book (LOB), see e.g. [PSS11, ASS12, LS13], and considers temporary impact to be transient, in the sense that it is persistent but decays over time. In most articles on transient price impact the impact is additive, instead of multiplicative as e.g. in [Løk12, BBF16]; A conceptual drawback is that then negative asset prices can occur with (small) positive probability. The article [Kyl85] distinguishes depth, resilience and tightness as key characteristics of liquidity. Depth corresponds to the (current) shape of the LOB. Finite resilience corresponds to the extent and swiftness of transience in impact. Tightness is commonly reflected by closeness of bid-ask spreads; yet, our optimal liquidation problem will require only one side of the LOB, being posed over monotone strategies.

A key feature of the present paper is that resilience of prices is stochastic, in the sense that temporary price perturbations by large trades revert in a transient fashion back towards fundamental prices according to stochastic dynamics of Ornstein-Uhlenbeck (OU) type. More specifically, the price process $S = (S_t)_{t \geq 0} = (f(Y_t)\bar{S}_t)_{t \geq 0}$ observed in the market deviates by a factor $f(Y)$ from a fundamental price process \bar{S} which would prevail in absence of large traders. The price impact function f is positive and increasing; it corresponds to a (general) density shape of the LOB, cf. [BBF16, Sect.2.1]. The market impact process Y (see (2.6)) is an OU-process controlled by the large trader's strategy and can be viewed as a LOB volume effect process [PSS11]. We note that relative price impact $\Delta\theta_t \mapsto f(Y_{t-} + \Delta\theta_t)/f(Y_{t-})$ by current trades, and hence the depth currently offered by the LOB, can vary stochastically with Y as well, as the impact function f is not required to be exponential. In contrast, the optimal execution literature has predominantly studied deterministic or even constant parameterizations for depth and resilience; To analyze how certain liquidity characteristics affect optimal behavior it is, of course, sensible to keep other parts simple, cf. [LS13]. For stochastic illiquidity, we are aware of only one recent contribution [FSU15] with analytical results in continuous time for transient impact. [FSU15] study stochastic height (depth) of a block-shaped LOB in an additive impact model. They investigate mainly whether the state space divides into one action and inaction region (or more, non-intuitively) that are separated by a boundary; apart from existence no explicit description of the boundary is obtained. Let us mention that instantaneous stochastic impact is studied in [Alm12], and stochastic drift of the fundamental price is studied in [LS13] for transient additive impact in a block-shaped LOB. Stochastic LOB shapes could also arise if one attempts to rewrite a multiplicative LOB model as a (more complex) additive LOB model with additional state dependencies, cf. [Løk12, Sect.5].

For our multiplicative transient impact model with stochastic resilience, we take the fundamental price \bar{S} to be an exponential Brownian motion and permit for non-zero correlation with Y . In this setup, we study the optimal liquidation problem for infinite time horizon as a singular stochastic control problem of finite fuel type. Our control

objective (see (2.7)) involves terms like in [Tak97, DZ98, DM04] which depend explicitly on (\bar{S}, Y) with a summation of integrals over any jump of the control, while key aspects differ. [DM04] prove general existence of a (weak) optimal control for finite horizon, but do not obtain its structural properties. We construct an explicit solution to the singular control problem. The optimal strategy is given by the local time process of an obliquely reflected diffusion on a curved boundary in \mathbb{R}^2 , the state space being the impact process Y and the holdings in the risky asset. Optimal liquidation strategies in most of the literature are deterministic or static. In contrast to notable research on adaptive strategies (in different models) by [SS09, LA11], the stochasticity of our strategy arises from its adaptivity to the transient component of the price dynamics and is of local time type. To construct a candidate solution, we first restrict the set of optimization strategies to those described by reflected diffusions on monotone boundaries, and optimize over the set of possible boundaries. To be able to apply calculus of variations methods, we prove an explicit formula for the Laplace transform of the inverse local times of diffusions reflected on elastic boundaries, which retract according to the local time that the reflected process has spent on the boundary. Having derived a (one-sided) optimal boundary (Theorem 5.6), we construct a candidate value function as a classical solution to the HJB variational inequality (3.3) for the control problem; Optimality is verified by a martingale optimality principle (Proposition 3.2).

Reflected diffusions at elastic boundaries are an interesting topic in itself. Hence we study them in more generality, beyond the OU setup of the control problem. There is a rich literature on the Skorokhod problem of reflected diffusions in time-independent domains, going back to [Tan79] for normal reflection in convex domains. [DI93] study oblique reflection in non-smooth domains. [NÖ10] generalize the problem to oblique reflection in time-dependent domains and provide an extensive literature review, to which we refer. We investigate one-dimensional diffusions reflected at (monotone) boundaries that vary with local time. Our main contribution (Theorem 7.3) here is the Laplace transform of the inverse local time, derived by a constructive approximation of the reflected diffusion. A byproduct of the construction is existence (cf. Remark 7.4).

Solutions to singular control problems are typically described by reflected diffusions. In seminal examples from literature, e.g. on the monotone follower problem and its generalizations, cf. [BSW81, KS86, KOWZ00], where such solutions to singular control problems are derived explicitly, the controlled process is the sum of a Brownian motion and the control, $W_t + \Theta_t$, and the optimal control is again obtained by reflecting the Brownian motion at an (elastic) boundary. In this particular case, the local time process has a (simple) direct representation as a running maximum due to the Brownian dynamics of the controlled process (yet, cf. [SS89] for a notable exception where the existence of an optimally controlled two-dimensional Brownian motion is proved without explicitly constructing the reflecting boundary). In contrast, for our application we consider a mean-reverting process of Ornstein-Uhlenbeck type driven by a Brownian motion and a control (see (2.6)), which does not permit for such direct representation. The verification of optimality is complicated by the implicit nature of the eigenfunctions for the OU generator, which are given in terms of special (Hermite) functions.

The main contributions of the present paper are the following. We solve explicitly a two-dimensional singular control problem and characterize the optimal control, which

is stochastic and given by the local time process of a reflected Ornstein-Uhlenbeck process at a non-constant elastic boundary, that is described through an explicit ODE, see Theorem 3.1. To this end, we investigate (approximations of) SDEs with reflections at elastic boundaries and derive the representation (7.12) for the Laplace transform of the inverse local time at the boundary. Concerning the application, we solve an optimal liquidation problem in a novel model with stochastic illiquidity, where price impact is multiplicative, transient and where the volume effect process exhibits own stochastic noise, i.e. under stochastic resilience of actual prices S towards fundamental prices \bar{S} .

The paper is organized as follows. Section 2 formulates the singular stochastic control problem. Section 3 states the solution and gives an overview on the general course of arguments to come. In Section 4 a calculus of variations problem is posed, by restricting to strategies given by diffusions reflected at smooth boundaries. This builds on results on Laplace transforms of inverse local times of reflected diffusions at elastic boundaries from Section 7, which is presented in a way so that it could be read independently. The free boundary is thereby constructed in Section 5. By solving the HJB variational inequality (3.3), the value function and the optimal control are derived in Section 6.

2 The model and the optimal control problem

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with two correlated Brownian motions W and B with correlation coefficient $\hat{\rho} \in [-1, 1]$, such that we have

$$[W, B]_t = \hat{\rho}t, \quad t \geq 0. \quad (2.1)$$

for the quadratic co-variation of W and B . The filtration $(\mathcal{F}_t)_{t \geq 0}$ is assumed to satisfy the usual conditions of completeness and right continuity, so we can take RCLL versions for semimartingales. We refer to [JS03] for notions from stochastic analysis.

We consider a market with a risky asset, in addition to the riskless numeraire asset whose (discounted) price is constant at 1. The large investor holds $\Theta_t \geq 0$ shares of the risky asset at time t . He may liquidate his initial position of Θ_{0-} shares by trading according to

$$\Theta_t := \Theta_{0-} - A_t, \quad (2.2)$$

where A is a predictable, càdlàg, monotone process, describing the cumulative number of assets sold up to time t . We define the set of admissible strategies A as

$$\begin{aligned} \mathcal{A}(\Theta_{0-}) := \{A \mid A \text{ non-decreasing, càdlàg, previsible,} \\ \text{with } 0 =: A_{0-} \leq A_t \leq \Theta_{0-}\}. \end{aligned} \quad (2.3)$$

The unaffected fundamental price $\bar{S} = (\bar{S}_t)_{t \geq 0}$ of the risky asset evolves according to

$$d\bar{S}_t = \mu\bar{S}_t dt + \sigma\bar{S}_t dW_t, \quad S_0 \in (0, \infty), \text{ with } \sigma > 0, \mu \in \mathbb{R}, \quad (2.4)$$

as a geometric Brownian motion, in absence of perturbations by large investor trading. By trading, however, the large investor has market impact on the actual price

$$S_t := f(Y_t)\bar{S}_t, \quad (2.5)$$

of the risky asset through some impact process Y , by an increasing positive function $f > 0$ in C^1 with $f(0) = 1$. The process Y can be interpreted as a volume effect process, representing the transient volume displacement by large trades in a shadow limit order book (LOB) whose shape corresponds to the price impact function f , cf. [PSS11] or [BBF16, Sect. 2.1]. The effect from perturbations $dB_t - dA_t$ on the process

$$dY_t = -h_Y(Y_t) dt + dB_t - dA_t, \quad Y_{0-} = y, \quad (2.6)$$

with $h_Y(0) = 0$ and $h'_Y > 0$ are transient over time, in that Y is mean reverting towards zero. Sometimes we shall write $Y^{y,A}$ to stress the dependence of Y on its initial state y and the strategy A . For linear $h_Y(y) := \beta y$, $\beta > 0$, dynamics of Y are of Ornstein-Uhlenbeck type, driven by $dB - dA$. The stochastic noise herein constitutes own stochasticity for the transient evolution of market impact; one may interpret dB_t as aggregated transient influence on dY_t by other (uninformed) large ‘noise traders’.

For $\gamma \geq 0$, the γ -discounted proceeds up to time t from a liquidation strategy A are

$$L_t(y; A) := \int_0^t e^{-\gamma u} f(Y_u) \bar{S}_u dA_u^c + \sum_{\substack{0 \leq u \leq t \\ \Delta A_u \neq 0}} e^{-\gamma u} \bar{S}_u \int_0^{\Delta A_u} f(Y_u - x) dx, \quad t \geq 0, \quad (2.7)$$

where $A_t = A_t^c + \sum_{u \leq t} \Delta A_u$ is the (pathwise) decomposition of A into its continuous and pure jump part, and $Y = Y^{y,A}$ is given by (2.6). Jump terms in (2.7) can be explained by an LOB perspective or from stability considerations, cf. [BBF15, Sections 2.1, 6].

As L is increasing, the limit $L_\infty := \lim_{t \rightarrow \infty} L_t$ exists. The large trader’s optimization problem is to maximize expected (discounted) proceeds over an infinite time horizon

$$\max_{A \in \mathcal{A}(\Theta_{0-})} \mathbb{E}[L_\infty(y; A)] \quad \text{with} \quad v(y, \theta) := \sup_{A \in \mathcal{A}(\theta)} \mathbb{E}[L_\infty(y; A)], \quad (2.8)$$

where $v(y, \theta)$ denotes the value function for $y \in \mathbb{R}$ and $\theta \in [0, \infty)$.

Remark 2.1. The value function v is increasing in y and θ . Indeed, monotonicity in θ follows from $\mathcal{A}(\theta_1) \subset \mathcal{A}(\theta_2)$ for $\theta_1 \leq \theta_2$. For monotonicity in y , note that for $y_1 \leq y_2$ and any strategy $A \in \mathcal{A}(\theta)$ one has $Y_t^{y_1, A} \leq Y_t^{y_2, A}$ for all t , implying $L_t(y_1; A) \leq L_t(y_2; A)$.

For Sections 3–6, the functions f, h and the scalars $\mu, \gamma, \sigma, \hat{\rho}$ satisfy the conditions of

Assumption 2.2. C1. The resilience function $h_Y(y) = \beta y$ in (2.6) is linear with a constant positive rate of resilience $\beta \in (0, \infty)$.

C2. The impact function $f \in C^3(\mathbb{R})$ satisfies $f, f' > 0$ and $(f'/f)' < (\Phi'/\Phi)'$ where $\Phi := \Phi_\delta$ is the (up to a constant factor) unique positive and increasing solution to the ODE $\delta \Phi = \frac{1}{2} \sigma^2 \Phi'' + (\sigma \hat{\rho} - h_Y) \Phi'$, that is given in (4.9).

C3. The impact function f furthermore satisfies $(f'/f)' < (\Phi''/\Phi)'$.

C4. Moreover, the function $\lambda(y) := f'(y)/f(y)$, $y \in \mathbb{R}$, is bounded, i.e. there is $\lambda_{\max} \in (0, \infty)$ such that $0 < \lambda(y) \leq \lambda_{\max}$ for all $y \in \mathbb{R}$.

C5. $\delta := \gamma - \mu > 0$, that means the drift coefficient $-\delta \bar{S}$ for the γ -discounted fundamental price $e^{-\gamma t} \bar{S}_t$ is negative.

C6. The function $k(y) := \frac{1}{2} \frac{f''(y)}{f(y)} - (\beta + \delta) + (\sigma \hat{\rho} - \beta y) \frac{f'(y)}{f(y)}$ is strictly decreasing.

C7. There exists y_0 and $y_\infty \in \mathbb{R}$ where $f'/f = \Phi'/\Phi$ respectively $f'/f = \Phi''/\Phi'$ holds.

Assumption 2.2 is satisfied by e.g. $f(y) = \exp(\lambda y)$ with $\lambda \in (0, \infty)$, cf. Lemma 5.1 below; see [BBF16, Section 2.1] for the shape of the related multiplicative LOB.

Assumption C1 is needed twice: to pinpoint the special function Φ in Assumption C2 as a Hermite function, for which we can prove certain Turan-like inequalities in Lemma 5.1, that are crucial – together with C2 and C3 – to obtain a well-behaved free boundary in Lemma 5.3. Moreover, C1 and C6 are needed for our verification arguments, see the proof of Lemma 6.6. Assumptions C2 and C3 ensure uniqueness of the (boundary) points y_0 and y_∞ from Assumption C7 which are needed in Lemma 5.3. While C3, uniqueness of y_∞ , is not crucial there, it will be needed in (6.17) for the verification. The bound on λ in Assumption C4 is used to show some growth condition on the value function in Lemma 6.4, that is required to apply the martingale optimality principle (Proposition 3.2). The overall negative drift in Assumption C5 ensures that the optimization problem on an infinite time horizon has a finite value.

3 The optimal control and how it will be derived

This section states the main theorem which describes the solution to the singular stochastic control problem, and outlines afterwards the general course of arguments for proving it in the subsequent sections. To explain ideas, let us first motivate how the variational inequality (3.3), being the dynamical programming equation for the optimization problem at hand, is readily suggested by an application of the martingale optimality principle. To this end, consider for an admissible strategy A the process

$$G_t(y; A) := L_t(y; A) + e^{-\gamma t} \bar{S}_t \cdot V(Y_t, \Theta_t), \quad (3.1)$$

where $V \in C^{2,1}(\mathbb{R} \times [0, \infty); [0, \infty))$ is some function and $G_{0-}(y; A) = \bar{S}_0 V(Y_{0-}, \Theta_{0-})$. Suppose V can be chosen such that G is a supermartingale. Then one should have

$$\begin{aligned} \bar{S}_0 V(y, \Theta_{0-}) = \mathbb{E}[G_{0-}(y; A)] &\geq \lim_{T \rightarrow \infty} \mathbb{E}[L_T(y; A)] + \lim_{T \rightarrow \infty} e^{-\gamma T} \mathbb{E}[\bar{S}_T V(Y_T, \Theta_T)] \\ &= \mathbb{E}[L_\infty(y; A)] \end{aligned}$$

heuristically, provided that the second summand on the right-hand side converges to 0. Hence, for V being such that G is a supermartingale for every admissible strategy A and a martingale for at least one strategy A^* , one can conclude that V is essentially the value function for (2.8) (modulo the factor \bar{S}_0). To describe V , one may apply Itô's

formula to get

$$\begin{aligned}
dG_t = e^{-\gamma t} \bar{S}_t & \left(V_y(Y_{t-}, \Theta_{t-}) dB_t + \sigma V(Y_{t-}, \Theta_{t-}) dW_t \right. \\
& + ((\mu - \gamma)V + (\sigma \hat{\rho} - \beta Y_{t-})V_y + \tfrac{1}{2}V_{yy})(Y_{t-}, \Theta_{t-}) dt \\
& + (f - V_y - V_\theta)(Y_{t-}, \Theta_{t-}) dA_t^c \\
& \left. + \int_0^{\Delta A_t} (f - V_y - V_\theta)(Y_{t-} - x, \Theta_{t-} - x) dx \right). \tag{3.2}
\end{aligned}$$

Define, with $\delta = \gamma - \mu$, a differential operator on $C^{2,0}$ functions φ by

$$\mathcal{L}\varphi(y, \theta) := \frac{1}{2}\varphi_{yy}(y, \theta) + (\sigma \hat{\rho} - \beta y)\varphi_y(y, \theta) - \delta\varphi(y, \theta).$$

By equation (3.2), solving the Hamilton-Jacobi-Bellman (HJB) variational inequality

$$0 = \max\{f - V_y - V_\theta, \mathcal{L}V\} \quad \text{with boundary condition } V(y, 0) = 0, y \in \mathbb{R}, \tag{3.3}$$

would suffice for G to be a local (super-)martingale. This suggests the existence of a *sell region* \mathcal{S} (action region) where the dA -integrand $f - V_y - V_\theta$ is zero and it is optimal to trade (i.e. sell), and a *wait region* \mathcal{W} (inaction region) in which the dt -integrand $\mathcal{L}V$ is zero and it is optimal not to trade. Assume that the two regions

$$\mathcal{S} = \{(y, \theta) \in \mathbb{R} \times (0, \infty) \mid y(\theta) < y\} \quad \text{and} \quad \mathcal{W} = \{(y, \theta) \in \mathbb{R} \times (0, \infty) \mid y < y(\theta)\}$$

are separated by a free boundary $y(\theta)$. An optimal strategy, i.e. a strategy for which G is a martingale, would be described as follows: if $(Y_{0-}, \Theta_{0-}) \in \mathcal{S}$, perform a block sell of size ΔA_0 such that $(Y_0, \Theta_0) = (Y_{0-} - \Delta A_0, \Theta_{0-} - \Delta A_0) \in \partial\mathcal{S}$. Thereafter, sell just enough as to keep the process (Y, Θ) within $\overline{\mathcal{W}}$. In this way, the process (Y, Θ) should be described by a diffusion process that is reflected at the boundary $\partial\mathcal{W} \cap \partial\mathcal{S}$ in direction $(-1, -1)$, i.e. there is waiting in the interior and selling at the boundary, until all shares are sold at $\{(y, 0) \mid y < y(0)\} = \partial\mathcal{W} \setminus \partial\mathcal{S}$. For such reflected diffusions, Section 7 provides existence, uniqueness and important characteristics which are key to the subsequent construction of the optimal control. The solution of the optimal liquidation problem is indeed described by the local time process of a reflected diffusion along a boundary explicitly given by an ODE. This main result is stated as

Theorem 3.1. *Let Assumption 2.2 be satisfied. Then the ordinary differential equation*

$$y'(\theta) = \left(\frac{1}{\Phi} \frac{((\Phi')^2 - \Phi\Phi'')(f'\Phi' - f\Phi'')}{f'' \cdot (\Phi\Phi'' - (\Phi')^2) + f' \cdot (\Phi'\Phi'' - \Phi\Phi''') + f \cdot (\Phi'\Phi''' - (\Phi'')^2)} \right)(y(\theta))$$

with initial condition $y(0) = y_0$ admits a unique classical solution $y : [0, \infty) \rightarrow \mathbb{R}$. The function y is strictly decreasing and maps $[0, \infty)$ bijectively onto $(y_\infty, y_0]$. The optimal liquidation problem (2.8) admits a unique optimal strategy A^ that is characterized by the boundary y as follows:*

1. *If $Y_{0-} \geq y_0 + \Theta_{0-}$, sell everything immediately at time 0 and stop trading.*

2. Otherwise, if $y(\Theta_{0-}) < Y_{0-} < y_0 + \Theta_{0-}$, perform at time 0 a block trade of size $A_0^* := \Delta > 0$ so that $Y_{0-} - \Delta = y(\Theta_{0-} - \Delta)$, i.e. $Y_0 = Y_{0-} - \Delta$ is on the boundary y . Afterwards, sell as much as to keep with the least effort the impact process Y in the region \bar{W} until the asset position is liquidated (at some time τ). More precisely, the optimal strategy is given by $A^* := (\Delta + K)\mathbb{1}_{[0, \tau]}$, for (Y, K) being the unique continuous adapted processes with non-decreasing K , starting in $(Y_0, 0)$, which solve (in the sense of Theorem 7.3) the y -reflected SDE

$$\begin{aligned} Y_t &\leq y(\Theta_{0-} - \Delta - K_t), \\ dY_t &= (\sigma\hat{\rho} - \beta Y_t) dt + dB_t - dK_t, \\ dK_t &= \mathbb{1}_{\{Y_t = y(\Theta_{0-} - \Delta - K_t)\}} dK_t, \end{aligned}$$

on time interval $\llbracket 0, \tau \rrbracket$ with $\tau := \inf\{t \geq 0 \mid K_t = \Theta_{0-} - \Delta\}$.

The Laplace transform of the inverse local time $\tau_\ell := \inf\{t > 0 \mid K_t > \ell\}$ in case 2. is

$$\mathbb{E}[e^{-\lambda\tau_\ell}] = \frac{\Phi_\lambda(Y_0)}{\Phi_\lambda(y(\Theta_0))} \cdot \exp\left(\int_0^\ell (y'(\Theta_0 - a) + 1) \frac{\Phi'_\lambda(y(\Theta_0 - a))}{\Phi_\lambda(y(\Theta_0 - a))} da\right) \quad (3.4)$$

for $\lambda > 0$, $0 \leq \ell \leq \Theta_0 = \Theta_{0-} - \Delta$. In particular, time to liquidation τ is a.s. finite in any case. The function V constructed in Section 6 (cf. (6.7)) is a (classical) solution to the HJB variational inequality (3.3). The value function for the optimal control problem (2.8) is given by $v(y, \theta) = \bar{S}_0 V(y, \theta)$ for $y = Y_{0-} \in \mathbb{R}$ and $\theta \in [0, \infty)$.

In the sequel, we will find the value function for our stochastic control problem by constructing a classical solution of the variational inequality (3.3). Provided that the key variational inequalities for the (candidate) solution can be verified, optimality can then be verified by typical martingale arguments, see Proposition 3.2. Based on results on reflected diffusions from Theorem 7.3, we reformulate in Section 4 the optimization problem as a (nonstandard) calculus of variations problem. Its solution, derived in Section 5, provides a candidate for the free boundary, separating the regions of action and inaction, together with the value function on that boundary. Moreover, we show a (one-sided) local optimality property of the derived boundary (cf. Theorem 5.6). This will be crucial in Section 6 (cf. proof of Lemma 6.6) to verify (3.3) for the candidate value function, constructed there, in order to finally conclude the proof on p.24.

Proposition 3.2 (Martingale optimality principle). *Let $V : \mathbb{R} \times [0, \infty) \rightarrow [0, \infty)$ be a $C^{2,1}$ function with the following properties:*

1. For every $\Theta_{0-} \geq 0$, there exists constants C_1, C_2 so that

$$V(y, \theta) \leq C_1 \exp(C_2 y) \vee 1 \quad \text{for all } (y, \theta) \in \mathbb{R} \times [0, \Theta_{0-}];$$

2. For every $\Theta_{0-} \geq 0$ and $A \in \mathcal{A}(\Theta_{0-})$, the process G_t from (3.1) is a supermartingale, where $Y = Y^{y, A}$ is defined in (2.6), and additionally $G_0(y; A) \leq G_{0-}(y; A)$.

Then we have $\bar{S}_0 \cdot V(y, \theta) \geq v(y, \theta)$. Moreover, if there exists $A^* \in \mathcal{A}(\Theta_{0-})$ such that $G(y; A^*)$ is a martingale and $G_0(y; A^*) = G_{0-}(y; A^*)$ holds, then $\bar{S}_0 V(y, \theta) = v(y, \theta)$ and $v(y, \theta) = \mathbb{E}[L_\infty(y; A^*)]$ for $\Theta_{0-} = \theta \geq 0$.

Proof. By the supermartingale property we have for every $T \geq 0$

$$\begin{aligned} \bar{S}_0 V(Y_{0-}, \Theta_{0-}) &\geq \mathbb{E}[G_0(y; A)] \geq \mathbb{E}[L_T(y; A) + e^{-\gamma T} \bar{S}_T V(Y_T, \Theta_T)] \\ &= \mathbb{E}[L_T(y; A)] + e^{-\gamma T} \mathbb{E}[\bar{S}_T V(Y_T, \Theta_T)] \\ &= \mathbb{E}[L_T(y; A)] + e^{-\delta T} \bar{S}_0 \mathbb{E}[\mathcal{E}(\sigma W)_T V(Y_T, \Theta_T)]. \end{aligned} \quad (3.5)$$

By monotone convergence, the first summand above tends to $\mathbb{E}[L_\infty(y; A)]$ for $T \rightarrow \infty$. To see that the second summand converges to 0, consider the Ornstein-Uhlenbeck process $dX_t = -\beta X_t dt + dB_t$, $X_0 = y$. Applying Itô's formula shows that

$$e^{\beta t}(Y_t - X_t) = \int_{[0, t]} e^{\beta u} d\Theta_u \quad \forall t \geq 0.$$

Since Θ is non-increasing, we conclude that $Y_t \leq X_t$ for all $t \geq 0$. Let $p, q > 1$ be conjugate, i.e. $1 = 1/q + 1/p$. Using Hölder's inequality, one gets

$$\begin{aligned} \mathbb{E}[\mathcal{E}(\sigma W)_T V(Y_T, \Theta_T)] &\leq \mathbb{E}[\mathcal{E}(\sigma W)_T^p]^{1/p} \mathbb{E}[V(Y_T, \Theta_T)^q]^{1/q} \\ &= \mathbb{E}[\exp(p\sigma W_T - \frac{1}{2}p\sigma^2 T)]^{1/p} \mathbb{E}[V(Y_T, \Theta_T)^q]^{1/q} \\ &= \mathbb{E}[\mathcal{E}(p\sigma W)_T]^{1/p} \exp\left(\frac{1}{p}\left(\frac{1}{2}p^2\sigma^2 T - \frac{1}{2}p\sigma^2 T\right)\right) \mathbb{E}[V(Y_T, \Theta_T)^q]^{1/q} \\ &= \exp\left(\frac{p-1}{2}\sigma^2 T\right) \mathbb{E}[V(Y_T, \Theta_T)^q]^{1/q} \leq \exp\left(\frac{p-1}{2}\sigma^2 T\right) \mathbb{E}[C_1 \exp(qC_2 Y_T) \vee 1]^{1/q} \\ &\leq \exp\left(\frac{p-1}{2}\sigma^2 T\right) \mathbb{E}[C_1 \exp(qC_2 X_T) \vee 1]^{1/q}. \end{aligned}$$

Using the fact that X is a Gaussian process with mean $\mathbb{E}[X_T] = ye^{-\beta T}$ and variance $\text{Var}(X_T) = \frac{1}{2\beta}(1 - e^{-2\beta T})$, one obtains for $K := \mathbb{E}[C_1 \exp(qC_2 X_T) \vee 1]$ that

$$K \leq 1 + C_1 \exp\left(qC_2 \mathbb{E}[X_T] + \frac{1}{2}q^2 C_2^2 \text{Var}(X_T)\right) \leq 1 + C_1 \exp\left(qC_2 y + \frac{1}{4\beta}q^2 C_2^2\right).$$

This bound on K is independent of T . By choosing $p > 1$ such that $\frac{p-1}{2}\sigma^2 < \delta$, one gets that $\exp(-\delta T) \exp(\frac{p-1}{2}\sigma^2 T)$ is exponentially decreasing in T , and can conclude that the second summand in (3.5) converges to 0 for $T \rightarrow \infty$. This implies that $\bar{S}_0 V(y, \theta) \geq \mathbb{E}[L_\infty(y; A)]$ for all $A \in \mathcal{A}(\theta)$ and yields the first part of the claim. The second part follows similarly by noting that, if $A^* \in \mathcal{A}(\theta)$ is such that $G(y; A^*)$ is a martingale and $G_0(y; A) = G_{0-}(y; A)$, then we have equalities instead of inequalities in the estimates leading to (3.5). By taking $T \rightarrow \infty$ we conclude that $\bar{S}_0 V(y, \theta) = \mathbb{E}[L_\infty(y; A^*)]$. Since $\bar{S}_0 V(y, \theta) \geq v(y, \theta)$ by the first part of the claim, we deduce the optimality of A^* . \square

4 Reformulation as a calculus of variations problem

In this section we will restate the free boundary problem of the variational inequality (3.3) as a (nonstandard, at first) calculus of variations problem.

To sketch the idea and nonstandard features of the problem, suppose that the large trader has to liquidate $\Theta_0 \geq 0$ shares and that (Y_0, Θ_0) is already on the free boundary (after an initial jump). Suppose that the boundary below level Θ_0 is parametrized by a suitable function $g : [0, \Theta_0] \rightarrow \mathbb{R}$, i.e. the boundary up to level Θ_0 is given by $\{(g(\theta), \Theta_0 - \theta) \mid \theta \in [0, \Theta_0]\}$, so $g(0) = Y_0$. In this case, the expected proceeds corresponding to a strategy A^g , that sells only when the state process (Y, A^g) is on the boundary g , can be expressed in terms of the Laplace transform of the inverse local time of a diffusion reflected at the boundary, namely (Y, A^g) , see (4.7) below. Thus, the boundary g should be obtained as a maximizer of these proceeds and be characterized by methods from calculus of variations. To do so, we need however to overcome some original twists in our problem. In fact, no point (neither $g(0)$ nor $g(\Theta_0)$) of our desired free boundary is given from the outset, in contrast to the standard setup for (isoperimetric) calculus of variations problems. Moreover, the integrand in the cost functional (4.13) exhibits path-dependence, as it depends on the (Laplace transform of the) time that the reflected process has been spending on excursions. But using Section 7, one can reparameterize the problem (via (4.14)) to get a tractable calculus of variations problem (4.15)–(4.16), whose solution should describe the free boundary.

In what follows, we will assume at first (but justify later) that g satisfies the assumptions of Theorem 7.3 (i.e. $g \in C^1$ with $g' \geq 0$) to develop these ideas. Let τ_{Θ_0} be the stopping time when $A^g = \Theta_0$ under the measure \mathbb{P} . For the continuous strategy $A^g = A$ we have by [DM82, Theorem 57] for any $T \in [0, \infty)$, that $\mathbb{E}[L_T]$ equals

$$\mathbb{E}\left[\int_0^{\tau_{\Theta_0} \wedge T} f(Y_t) e^{-\delta t} \mathcal{E}(\sigma W)_t dA_t\right] = \mathbb{E}\left[\mathcal{E}(\sigma W)_T \int_0^{\tau_{\Theta_0} \wedge T} f(Y_t) e^{-\delta t} dA_t\right]. \quad (4.1)$$

For fixed T , let \mathbb{Q} be the measure given by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(\sigma W)_T$ on \mathcal{F}_T . Then

$$\mathbb{E}[L_T] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_{\Theta_0} \wedge T} f(Y_t) e^{-\delta t} dA_t\right]. \quad (4.2)$$

Girsanov's theorem gives that under \mathbb{Q} , the process $\tilde{B}_t := B_t - [B, \sigma W]_t = B_t - \sigma \hat{\rho} t$ is a Brownian motion. Therefore, we have under \mathbb{Q}

$$dY_t = (\sigma \hat{\rho} - \beta Y_t) dt + d\tilde{B}_t - dA_t, \quad (4.3)$$

i.e. the impact process Y is a (reflected) Ornstein-Uhlenbeck process with shifted (non-zero) mean reversion level, and A is its local time on the boundary. We cannot directly pass to the limit $T \rightarrow \infty$ in (4.2) because the measure change \mathbb{Q} depends on T . However, note that the right-hand side of (4.2) depends only on the law of the reflected diffusion (Y, A) under the measure \mathbb{Q} . That is why we consider the reflected diffusion

(X, A^X) with the following dynamics under the initial measure \mathbb{P} :

$$dX_t = (\sigma\hat{\rho} - \beta X_t) dt + dB_t - dA_t^X, \quad X_0 = g(0), \quad (4.4)$$

$$dA_t^X = \mathbb{1}_{X_t=g(A_t^X)} dA_t^X, \quad A_0^X = 0, \quad (4.5)$$

$$\tau_\ell^X = \inf\{t > 0 \mid A_t^X > \ell\}. \quad (4.6)$$

Note that the difference between the impact process Y and the process X is just the shift in the drift of X by $\sigma\hat{\rho}$, and hence we will occasionally refer to X as the *shifted impact process*. Now, by (4.2) we have $\mathbb{E}[L_T] = \mathbb{E}\left[\int_0^{\tau_{\Theta_0}^X \wedge T} f(X_t) e^{-\delta t} dA_t^X\right]$, which gives for $T \rightarrow \infty$ by monotone convergence on both sides

$$\begin{aligned} \mathbb{E}[L_\infty] &= \mathbb{E}\left[\int_0^{\tau_{\Theta_0}^X} f(X_t) e^{-\delta t} dA_t^X\right] = \mathbb{E}\left[\int_0^{\tau_{\Theta_0}^X} f(g(A_t^X)) e^{-\delta t} dA_t^X\right] \\ &= \mathbb{E}\left[\int_0^{\Theta_0} f(g(\ell)) e^{-\delta \tau_\ell^X} d\ell\right] = \int_0^{\Theta_0} f(g(\ell)) \mathbb{E}[e^{-\delta \tau_\ell^X}] d\ell \end{aligned} \quad (4.7)$$

$$= \int_0^{\Theta_0} f(g(\ell)) \exp\left(-\int_0^\ell (g'(a) + 1) \frac{\Phi'_\delta(g(a))}{\Phi_\delta(g(a))} da\right) d\ell, \quad (4.8)$$

using (4.5) and Theorem 7.3.

A simple corollary of Theorem 7.3 is that the inverse local time in the case of reflected Ornstein-Uhlenbeck process is almost surely finite and we can express its moments in terms of the Hermite functions. Note that, up to a positive multiplicative constant,

$$\Phi_\delta(x) = H_{-\delta/\beta}\left((\sigma\hat{\rho} - \beta x)/\sqrt{\beta}\right), \quad (4.9)$$

where $H_\nu(x)$ denotes the Hermite function with parameter ν , i.e. H_ν is a positive solution of the ODE $\varphi''(x) - 2x\varphi'(x) + 2\nu\varphi(x) = 0$ (cf. [Leb72, (10.2.8), p.285]), and Φ_δ from Assumption 2.2. Let us recall the following facts about Hermite functions that are needed in the remainder of the paper; we will concentrate on real parameters $\nu \in \mathbb{R}$, but similar statements hold for $\nu \in \mathbb{C}$, see the references given.

1. H is analytic both on the argument and the parameter, see [Leb72, p. 285].
2. The derivative is given [Leb72, (10.4.4), p. 289] by

$$H'_\nu(x) = 2\nu H_{\nu-1}(x). \quad (4.10)$$

3. If $\nu < 0$, Hermite functions have [Leb72, (10.5.2), p. 290] an integral representation

$$H_\nu(x) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-t^2 - 2xt} t^{-\nu-1} dt. \quad (4.11)$$

4. If $\nu < 0$, H_ν is strictly decreasing and $\lim_{x \rightarrow +\infty} H_\nu(x) = 0$ [Leb72, (10.6.4), p. 292].

Corollary 4.1. *Consider the setup from Theorem 7.3 with the reflected OU process X solving (4.4) – (4.5) with its inverse local time τ^X as defined in (4.6). Then for every $k \in \mathbb{N}$ the k -th moment of the inverse local time exists and is given by*

$$\mathbb{E}[(\tau_\ell^X)^k] = (-1)^k \frac{d}{d\delta} \Big|_{\delta=0} \exp\left(-\int_0^\ell (g'(a) + 1) \frac{\Phi'_\delta(g(a))}{\Phi_\delta(g(a))} da\right). \quad (4.12)$$

In particular, $\tau_\ell < \infty$ a.s. for every $\ell \geq 0$.

Proof. Note that equality (7.12) is trivially satisfied also for $\delta = 0$ since $\Phi_0(\cdot) = 1$. By the analyticity of the Hermite functions on the parameter, the right-hand side of (7.12), as a function of δ , is infinitely often differentiable at 0, because $\Phi_0(\cdot) = 1$, see also [Kle08, Thm. 6.28] for an argument of why we can exchange integration w.r.t. the variable a and differentiation w.r.t. δ . On the other hand, dominated convergence gives that these derivatives are exactly equal to $\mathbb{E}[(-\tau_\ell^X)^k]$ by considering the right-derivatives ($\delta \searrow 0$), see [Kle08, Example 6.29] for a more detailed account. \square

In the following we often abbreviate $\Phi := \Phi_\delta$. Since $g(a) = y(\Theta_0 - a)$ we get from (4.8) with $\rho(\ell) := \int_0^\ell (1 - y'(a)) \frac{\Phi'(y(a))}{\Phi(y(a))} da$ that

$$\mathbb{E}[L_\infty] = e^{-\rho(\Theta_0)} \int_0^{\Theta_0} f(y(\ell)) e^{\rho(\ell)} d\ell. \quad (4.13)$$

Since $\Phi', \Phi > 0$ and by assumption $y' < 0$, the function ρ is strictly increasing and so has an inverse ρ^{-1} . Fixing $R := \rho(\Theta_0)$ and setting $w(r) := y(\rho^{-1}(r))$, we find

$$\rho^{-1}(r) = \int_0^r \left(w'(z) + \frac{\Phi(w(z))}{\Phi'(w(z))} \right) dz. \quad (4.14)$$

Hence, finding a maximizing function y for (4.13) can be reduced to the problem of finding a function w which maximizes

$$J(w) := \int_0^R f(w(r)) e^{-(R-r)} \left(w'(r) + \frac{\Phi(w(r))}{\Phi'(w(r))} \right) dr \quad (4.15)$$

$$\text{with subsidiary condition} \quad K(w) := \int_0^R \left(w'(r) + \frac{\Phi(w(r))}{\Phi'(w(r))} \right) dr = \Theta_0. \quad (4.16)$$

The requested boundary function then is $y(\theta) = w(\rho(\theta))$.

5 Solving the calculus of variations problem

In this section, we solve the calculus of variations problem of maximizing (4.15) subject to (4.16) locally by employing necessary and sufficient conditions on the first and second variation of the functionals involved. We obtain the candidate free boundary $y(\theta)$, see equations (5.11) and (5.12), and show its local optimality in Lemma 5.4. We then relate our results on the calculus of variations problem to the initial optimal execution problem in Theorem 5.6. This will be crucial later for Section 6 to verify the desired inequality in the sell region, presented in Lemma 6.6. We will use notation and terminology from [GF00].

A maximizer w of the isoperimetric problem (4.15)–(4.16) also maximizes $J + mK$ for some constant $m = m(R)$ that is the Lagrange multiplier, cf. [GF00, Theorem 2.12.1]. The first variation $\delta(J + mK)$ vanishes if its corresponding Euler-Lagrange equation

$$0 = F_w - \frac{d}{dr} F_{w'} + m \cdot \left(G_w - \frac{d}{dr} G_{w'} \right) \quad (5.1)$$

holds, with $G(r, w, w') := w' + \Phi(w)/\Phi'(w)$ and $F(r, w, w') := f(w)e^{-(R-r)}G(r, w, w')$, the integrands of K and J , respectively. One side of the boundary is fixed $w(R) = y(\Theta_0)$, but the other side $w(0)$ is free, which gives rise to the natural boundary condition

$$0 = F_{w'} + mG_{w'} \Big|_{r=0}, \quad (5.2)$$

which yields $m(R) = -f(y_0)e^{-R}$ for $y_0 := y(0) = w(0)$. After multiplication with $e^R\Phi'(w)^2$, equation (5.1) simplifies to

$$e^r\Phi(w)(f'(w)\Phi'(w) - f(w)\Phi''(w)) = f(y_0)(\Phi'(w)^2 - \Phi(w)\Phi''(w)). \quad (5.3)$$

Inserting $r = 0$ gives a condition for y_0 , namely

$$f'(y_0)\Phi(y_0) = f(y_0)\Phi'(y_0). \quad (5.4)$$

Assumption C7 guarantees existence and C2 uniqueness of y_0 . On the other hand, differentiating both sides of (5.3) with respect to r gives an ODE for w , namely

$$\begin{aligned} 0 = w' \cdot (e^r(f'\Phi' - f\Phi'')\Phi' + e^r(f''\Phi' - f\Phi''')\Phi - f(y_0)(\Phi'\Phi'' - \Phi\Phi''')) \\ + e^r(f'\Phi' - f\Phi'')\Phi \end{aligned} \quad (5.5)$$

with abbreviations $f := f(w(r))$, $\Phi := \Phi(w(r))$, etc.

Both sides in the above equality (5.3) are negative on the boundary $w(r)$, due to

Lemma 5.1. *The positive, increasing eigenfunctions $\Phi = \Phi_\delta$ corresponding to the eigenvalue $\delta > 0$ of the generator of an Ornstein-Uhlenbeck process satisfy*

$$(\Phi^{(n)}(x))^2 < \Phi^{(n-1)}(x)\Phi^{(n+1)}(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In particular, $(\Phi')^2 < \Phi\Phi''$. Moreover, for $n \in \mathbb{N}$

$$\lim_{x \rightarrow -\infty} \Phi^{(n)}(x)/\Phi^{(n-1)}(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \Phi^{(n)}(x)/\Phi^{(n-1)}(x) = +\infty.$$

Proof. By equations (4.9) and (4.10), we have

$$\Phi_\delta^{(n)}\Phi_\delta^{(n+2)} - (\Phi_\delta^{(n+1)})^2 = (\Phi_{\delta+n\beta}\Phi_{\delta+n\beta}'' - (\Phi_{\delta+n\beta}')^2) \frac{2^{2n}}{\sigma^{2n}\beta^n} \prod_{k=0}^n (\delta + k\beta)^2,$$

so it suffices to prove $(\Phi')^2 < \Phi''\Phi$ for every $\delta, \beta, \sigma > 0$ and $\hat{\rho} \in [-1, 1]$ in (4.9). This is equivalent to showing $(H'_\nu)^2 < H''_\nu H_\nu$ for every $\nu < 0$. By (4.11) and since $\Gamma(-\nu) > 0$, the function $\varphi_x(t) := e^{-t^2-2xt}t^{-\nu-1}$ is the density of some absolutely continuous finite measure μ on $[0, \infty)$. For the probability measure $\tilde{\mathbb{P}}[A] := \mu([0, \infty))^{-1}\mu(A)$ consider two independent random variables $X, Y \sim \tilde{\mathbb{P}}$. By [Kle08, Thm. 6.28], we can exchange integration and differentiation, so $H''_\nu(x)H_\nu(x) - (H'_\nu(x))^2 = 4\tilde{\mathbb{E}}[X^2 - XY]$. Symmetry gives $2\tilde{\mathbb{E}}[X^2 - XY] = \tilde{\mathbb{E}}[(X - Y)^2] \geq 0$. Since X and Y are independent with absolutely continuous distribution, Fubini's theorem yields $\tilde{\mathbb{P}}[X = Y] = 0$, so $\tilde{\mathbb{E}}[(X - Y)^2] > 0$.

The asymptotic behavior of $\Phi^{(n)}/\Phi^{(n-1)}$ follows from [Leb72, (10.6.4) on p. 292] in the case $x \rightarrow -\infty$ and from [Leb72, (10.6.7) on p. 292] in the case $x \rightarrow +\infty$. \square

This gives a representation of r given y_0 and w as

$$r = \log \frac{f(y_0)}{\Phi(w)} + \log \frac{\Phi'(w)^2 - \Phi(w)\Phi''(w)}{f'(w)\Phi'(w) - f(w)\Phi''(w)}, \quad (5.6)$$

which we can use to simplify the ODE (5.5) (assuming $w' \neq 0$ everywhere) to

$$\frac{1}{w'} = -\frac{\Phi'}{\Phi} + \frac{f\Phi''' - f''\Phi'}{f'\Phi' - f\Phi''} + \frac{\Phi'\Phi'' - \Phi\Phi'''}{(\Phi')^2 - \Phi\Phi''}, \quad (5.7)$$

reading the right hand side as a function of $w(r)$. With $y(\theta) = w(\rho(\theta))$ and $r := \rho(\theta)$, we get $y'(\theta) = w'(r)\rho'(\theta) = w'(r)(1 - y'(\theta))\Phi'(y(\theta))/\Phi(y(\theta))$, which simplifies to

$$y'(\theta) = \frac{\Phi'(y)}{\Phi'(y) + \Phi(y)/w'(r)} \quad (5.8)$$

$$= \frac{1}{\Phi} \frac{((\Phi')^2 - \Phi\Phi'')(f'\Phi' - f\Phi'')}{f''(\Phi\Phi'' - (\Phi')^2) + f'(\Phi'\Phi'' - \Phi\Phi''') + f(\Phi'\Phi''' - (\Phi'')^2)} \quad (5.9)$$

$$= \frac{M_2(y(\theta))}{M_1'(y(\theta))} \quad (5.10)$$

$$\text{where } M_1 := \frac{f\Phi' - f'\Phi}{(\Phi')^2 - \Phi\Phi''} \quad \text{and} \quad M_2 := \frac{f'\Phi' - f\Phi''}{(\Phi')^2 - \Phi\Phi''}. \quad (5.11)$$

By (5.3) and Lemma 5.1 we have $M_2(y(\theta)) > 0$ for any θ . We get $M_1'(y(\theta)) < 0$ by

Lemma 5.2. *Assume f satisfies Assumption C2. Then $M_1'(y) < 0$ for all $y \in \mathbb{R}$.*

Proof. Let $G := \Phi'/\Phi$ and $H := \Phi''/\Phi'$. By Lemma 5.1, we have $G, G', H, H' > 0$ and $G < H$. With $\lambda(y) = f'(y)/f(y) > 0$, so $f''/f = \lambda' + \lambda^2$ and we get

$$(G')^2\Phi M_1'/f = \lambda'G' + (\lambda^2 - \lambda H)G' + (G^2 - \lambda G)H'.$$

So $M_1'(y) < 0$ if and only if $\lambda'(y)G'(y) < q(\lambda(y))$ where $q(\lambda) := (H - \lambda)\lambda G' + (\lambda - G)GH'$. The function q is quadratic in λ and takes its minimum in

$$\lambda^* := \frac{HG' + GH'}{2G'} \quad \text{with value} \quad q(\lambda^*) = \frac{(HG' + GH')^2}{4G'} - G^2H'.$$

Note also, that $G' = (H - G)G$. We find that

$$\begin{aligned} 4G'(\lambda'G' - q(\lambda)) &\leq 4G'(\lambda'G' - q(\lambda^*)) < 4G'((G')^2 - q(\lambda^*)) \\ &= 4(G')^3 - (GH' + G'H)^2 + 4G'G^2H' \\ &= G^2(4G(H - G)^3 - (H' + (H - G)H)^2 + 4(H - G)GH') \\ &= -G^2(H' + H^2 + 2G^2 - 3GH)^2 < 0, \end{aligned}$$

using that $\lambda'(y) < G'(y)$, $y \in \mathbb{R}$, by Assumption C2. So $M_1'(y) < 0$ for all $y \in \mathbb{R}$. \square

Lemma 5.3. *Let f satisfy Assumptions C2, C3 and C7. Then there exists a unique solution $\theta \mapsto y(\theta)$, $\theta \in [0, \infty)$, of the ODE*

$$y' = M_2(y)/M_1'(y), \quad y(0) = y_0, \quad (5.12)$$

and y is strictly decreasing with values in $(y_\infty, y_0]$ (with y_0 and y_∞ from Assumption C7).

Proof. Since M_2/M_1' is locally Lipschitz by $f \in C^3(\mathbb{R})$, there exists a unique maximal solution $y : [0, \theta_{\max}) \rightarrow \mathbb{R}$ of (5.12). We have $M_2(y(\theta)) > 0$ and $M_1' < 0$ by Lemma 5.2, thus $y' < 0$. Assume that $\theta_{\max} < \infty$, which therefore implies $\lim_{\theta \rightarrow \theta_{\max}} y(\theta) = -\infty$. However, note that $[0, \infty) \times \{y_\infty\}$ and $\{(\theta, y(\theta)) \mid 0 \leq \theta < \theta_{\max}\}$ are trajectories of the two-dimensional dynamical system induced by the field $(\theta, y) \mapsto (1, M_2(y)/M_1'(y))$. Since trajectories of dynamical systems cannot cross, and $y_\infty < y_0$ by Lemma 5.1, we must have $y_\infty < y(\theta)$ for all $\theta \in [0, \theta_{\max})$, which contradicts $\theta_{\max} < \infty$. \square

By considering the first variation $\delta(J + mK)$, we found a candidate boundary y by means of a possible extremum $w : [0, R] \rightarrow \mathbb{R}$ of $J + mK$. Calculating the second variation $\delta^2(J + mK)$ at w , we find that w is indeed a local maximizer.

For a C^1 -perturbation $h : [0, R] \rightarrow \mathbb{R}$ of w with $h(0) = h(R) = 0$ we have

$$\delta^2(J + mK)[w; h] = \int_0^R (Ph'(r)^2 + Qh(r)^2) dr \quad (5.13)$$

with $P = P(r, w(r), w'(r))$ and $Q = Q(r, w(r), w'(r))$ given by

$$P = \frac{1}{2}(F_{w'w'} + mG_{w'w'}) = 0, \quad (5.14)$$

$$Q = \frac{1}{2}\left(F_{ww} + mG_{ww} - \frac{d}{dr}\left(F_{ww'} + mG_{ww'}\right)\right) \quad (5.15)$$

$$= \frac{1}{2}e^{-(R-r)}\left(f'' \frac{\Phi}{\Phi'} + 2f' \cdot \left(\frac{\Phi}{\Phi'}\right)' + f \cdot \left(\frac{\Phi}{\Phi'}\right)'' - f'\right) + \frac{1}{2}m \cdot \left(\frac{\Phi}{\Phi'}\right)'', \quad (5.16)$$

with f, Φ and their derivatives being evaluated at $w(r)$ when no argument is mentioned. The d/dr -differentiation of (5.1) yields

$$\begin{aligned} 0 &= \frac{d}{dr}e^{-(R-r)}\left(f' \cdot \frac{\Phi}{\Phi'} + f \cdot \left(\frac{\Phi}{\Phi'}\right)' - f\right) + m \frac{d}{dr}\left(\frac{\Phi}{\Phi'}\right)' \\ &= e^{-(R-r)}\left(f' \cdot \frac{\Phi}{\Phi'} + f \cdot \left(\frac{\Phi}{\Phi'}\right)' - f\right) \\ &\quad + e^{-(R-r)}\left(f'' \frac{\Phi}{\Phi'} + 2f' \cdot \left(\frac{\Phi}{\Phi'}\right)' + f \cdot \left(\frac{\Phi}{\Phi'}\right)'' - f'\right)w' + m \cdot \left(\frac{\Phi}{\Phi'}\right)''w' \\ &= e^{-(R-r)}\left(f' \cdot \frac{\Phi}{\Phi'} + f \cdot \left(\frac{\Phi}{\Phi'}\right)' - f\right) + 2Qw' \\ &= e^{-(R-r)}\frac{\Phi}{(\Phi')^2}(f'\Phi' - f\Phi'') + 2Qw'. \end{aligned} \quad (5.17)$$

By equation (5.3) and Lemma 5.1, the first summand in (5.17) is negative along $w(r)$. Since $w(r) = y(\rho^{-1}(r))$ and ρ^{-1} is strictly increasing, we have $w' < 0$ by Lemma 5.3. So it must hold $Q(r, w(r), w'(r)) < -\kappa < 0$ on $[0, R]$ by (5.17) for some constant $\kappa = \kappa_R$, which gives that the second variation is negative definite at w ,

$$\delta^2(J + mK)[w; h] = \int_0^R Q(r, w(r), w'(r)) h(r)^2 dr < -\kappa \int_0^R h(r)^2 dr < 0, \quad (5.18)$$

for $h \neq 0$. Hence, by a classical calculus of variations argument we get the following

Lemma 5.4. *The functional $\hat{J} := J + mK : C^1([0, R]) \rightarrow \mathbb{R}$ defined by (4.15) – (4.16) with $m = -f(y_0)e^{-R}$ has a strict local maximizer $w(r) = y(\rho^{-1}(r))$ given by (5.12) in the following sense. There exists $\varepsilon > 0$ such that for all perturbations $0 \neq h \in C^1([0, R])$ with $h(0) = h(R) = 0$ and $\|h\|_{W^{1,\infty}} = \|h\|_\infty \vee \|h'\|_\infty < \varepsilon$ it holds*

$$\hat{J}(w + h) < \hat{J}(w).$$

Proof. To shorten notation, let $\hat{F} := F + mG$ so $\hat{J} := \int_0^R \hat{F} dr$. Unless the arguments are explicitly written, take $\hat{F} = \hat{F}(r, w(r), w'(r))$. Taylor's theorem gives $\hat{J}(w + h) - \hat{J}(w) = \delta \hat{J}[w; h] + \delta^2 \hat{J}[w; h] + \mathcal{E}(h)$ with first variation $\delta \hat{J}[w; h] = 0$ by (5.1), second variation $\delta^2 \hat{J}[w; h] = \int_0^R Q h^2 dr < 0$ by (5.18) and remainder

$$\mathcal{E}(h) = \int_0^R \left(\sum_{|\alpha|=3} \partial^\alpha \hat{F}(r, \mathbf{w} + \xi_r \mathbf{h}) \frac{\mathbf{h}^\alpha}{\alpha!} \right) dr$$

for some $\xi_r \in [0, 1]$, with $\mathbf{w} = (w(r), w'(r))^\top$, $\mathbf{h} = (h(r), h'(r))^\top$ and multi-index $\alpha \in \mathbb{N}_0^2$, considering $\hat{F}(r, \cdot)$ as an function on \mathbb{R}^2 . Since \hat{F} is linear in w' we get

$$\mathcal{E}(h) = \int_0^R \left(\frac{1}{6} \hat{F}_{www}(r, \mathbf{w} + \xi_r \mathbf{h}) \cdot h + \frac{1}{2} \hat{F}_{www'}(r, \mathbf{w} + \xi_r \mathbf{h}) \cdot h' \right) h^2 dr =: \int_0^R A \cdot h^2 dr$$

Note that by compactness of $[0, R]$ we have $A \rightarrow 0$ uniformly as $\|h\|_{W^{1,\infty}} \rightarrow 0$. Now choose $\varepsilon > 0$ small enough such that $|A| < \kappa/2$ for all $\|h\|_{W^{1,\infty}} < \varepsilon$, where $-\kappa < 0$ is an upper bound of Q . Hence, with $h \neq 0$

$$\hat{J}(w + h) - \hat{J}(w) = \int_0^R (Q + A) h^2 dr < -\frac{\kappa}{2} \int_0^R h^2 dr < 0. \quad \square$$

Note that the definition $w(r) := y(\rho^{-1}(r))$ does not depend on the interval boundary R . Hence the optimizer w over $[0, R]$ from Lemma 5.4 is optimal for all $R > 0$. We can calculate the value $J(w)$ of our optimizer explicitly.

Lemma 5.5. *For the optimal w from Lemma 5.4 we have*

$$J(w) = (\Phi M_1)(y(\Theta_0)) = (\Phi M_1)(w(R))$$

Proof. By direct calculation we have $\frac{fM'_1}{\Phi M_2^2} = \left(\frac{f\Phi' - f'\Phi}{f'\Phi' - f\Phi''}\right)'$. Moreover, (5.3) gives $e^r = f(y_0)/(\Phi M_2)(w(r))$. Substituting $r := \rho(\ell)$ to (4.13) and $y' = (M_2/M'_1)(y)$ yields

$$\begin{aligned} J(w) &= e^{-\rho(\Theta_0)} \int_0^{\Theta_0} f(y(\ell)) e^{\rho(\ell)} d\ell = (\Phi M_2)(y(\Theta_0)) \int_0^{\Theta_0} \left(\frac{f}{\Phi M_2}\right)(y(\ell)) d\ell \\ &= (\Phi M_2)(y(\Theta_0)) \int_{y_0}^{y(\Theta_0)} \left(\frac{fM'_1}{\Phi M_2^2}\right)(x) dx = (\Phi M_2)(y(\Theta_0)) \left[\frac{f\Phi' - f'\Phi}{f'\Phi' - f\Phi''}\right]_{y_0}^{y(\Theta_0)} \\ &= (\Phi M_1)(y(\Theta_0)). \quad \square \end{aligned}$$

To translate the results obtained so far back to the state space of impact and asset position, let us make the

Definition (\tilde{y} -reflected strategy). For a boundary function $\tilde{y} : [0, \infty) \rightarrow \mathbb{R}$ with $\tilde{y}' \leq 0$ and initial assets $\Theta_0 \geq 0$, denote by $A^{\text{ref}}(\tilde{y})$ the selling strategy for which the shifted impact process X starting at $\tilde{y}(\Theta_0)$ is continuous and reflected at \tilde{y} until all the shares are sold. More precisely, $A = A^{\text{ref}}(\tilde{y})$ is such that the (unique) pair (X, A) of continuous adapted processes with non-decreasing A solves (4.4)–(4.5) for boundary $g(\cdot) := \tilde{y}(\Theta - \cdot)$ defined on $[0, \Theta_0]$, until the liquidation time $\tau = \inf\{t > 0 \mid A_t \geq \Theta_0\}$.

Now we can formulate and prove the following result that will be crucial for our analysis in the verification argument in Section 6.

Theorem 5.6. *The function $y : [0, \infty) \rightarrow \mathbb{R}$ defined by equation (5.12) and y_0 is a (one-sided) local maximizer of $\mathbb{E}[L_\infty(y) \mid \Theta_0]$ in the sense that for every $\theta > 0$ there exists some $\varepsilon > 0$ such that for any decreasing $\tilde{y} \in C^1([0, \infty))$ with $y(\cdot) \leq \tilde{y}(\cdot) \leq y_0$, $y = \tilde{y}$ on $[\theta, \infty)$ and $0 < \|y - \tilde{y}\|_{W^{1,\infty}} < \varepsilon$ it holds*

$$\mathbb{E}[L_\infty(y) \mid \Theta_0 = \theta] > \mathbb{E}[L_\infty(\tilde{y}) \mid \Theta_0 = \theta].$$

Proof. For sake of clarity, we write $J = J_R$ and $K = K_R$ to emphasize the dependence of the functionals J, K on R . Call $w(r)$ the parametrization of y and $\tilde{w}(r)$ the parametrization of \tilde{y} .

Fix $\theta > 0$ and choose $R, \hat{R}, \hat{\theta}$ such that $y(\theta) = w(R)$, $\tilde{y}(\theta) = \tilde{w}(\hat{R})$ and $w(\hat{R}) = y(\hat{\theta})$. So $R := \rho_y(\theta)$, $\hat{R} := \rho_{\tilde{y}}(\theta) = \int_0^\theta \frac{\Phi'(\tilde{y}(x))}{\Phi(\tilde{y}(x))} dx + \int_{\tilde{y}(\theta)}^{\tilde{y}(0)} \frac{\Phi'(u)}{\Phi(u)} du$ and $\hat{\theta} := \rho_y^{-1}(\hat{R})$. By $y \neq \tilde{y}$, $y(\cdot) \leq \tilde{y}(\cdot)$ with equality outside $(0, \theta)$ and monotonicity of Φ'/Φ , we have $\hat{R} > R$ and thus $\hat{\theta} > \theta$.

Now, $K_{\hat{R}}(w) = \hat{\theta}$ and $K_{\hat{R}}(\tilde{w}) = \theta$. Moreover, $J_r(w) = (\Phi M_1)(w(r))$ by Lemma 5.5. So if $\|w - \tilde{w}\|_{W^{1,\infty}}$ is small enough, by Lemma 5.4 we get

$$\begin{aligned} J_R(w) &= (\Phi M_1)(w(R)) - (\Phi M_1)(w(\hat{R})) + J_{\hat{R}}(w) \\ &= (\Phi M_1)(w(R)) - (\Phi M_1)(w(\hat{R})) + e^{-\hat{R}} f(y_0) \hat{\theta} + (J_{\hat{R}} - e^{-\hat{R}} f(y_0) K_{\hat{R}})(w) \\ &> (\Phi M_1)(w(R)) - (\Phi M_1)(w(\hat{R})) + e^{-\hat{R}} f(y_0) \hat{\theta} + (J_{\hat{R}} - e^{-\hat{R}} f(y_0) K_{\hat{R}})(\tilde{w}) \\ &= (\Phi M_1)(y(\hat{\theta} - \eta)) - (\Phi M_1)(y(\hat{\theta})) + e^{-\hat{R}} f(y_0) \eta + J_{\hat{R}}(\tilde{w}) =: g(\eta) + J_{\hat{R}}(\tilde{w}). \end{aligned}$$

where $\eta := \hat{\theta} - \theta > 0$. By (5.6) we get $e^{-\hat{R}}f(y_0) = (\Phi M_2)(y(\hat{\theta}))$. With (5.10) follows

$$\begin{aligned} g'(\eta) &= -\left((\Phi M_1)' \frac{M_2}{M_1'}\right)(y(\hat{\theta} - \eta)) + (\Phi M_2)(y(\hat{\theta})) \\ &= -\left(\frac{\Phi' M_1 M_2}{M_1'} + \Phi M_2\right)(y(\hat{\theta} - \eta)) + (\Phi M_2)(y(\hat{\theta})) \end{aligned}$$

Hence $g'(0) = -(\Phi' M_1 M_2 / M_1')(y(\hat{\theta}))$. Since $M_1 > 0$ on $(-\infty, y_0)$, $M_2 > 0$ on $(y_\infty, y_0]$, $M_1' < 0$ by Lemma 5.2 and $\Phi' > 0$ it follows $g'(0) > 0$. So $g(\eta) > 0$ for $\eta > 0$ small enough. Hence it follows

$$\mathbb{E}[L_\infty(y) \mid \Theta_0 = \theta] = J_R(w) > J_{\hat{R}}(\tilde{w}) = \mathbb{E}[L_\infty(\tilde{y}) \mid \Theta_0 = \theta]$$

The bounds on η and $\|w - \tilde{w}\|_{W^{1,\infty}}$ are satisfied for small enough $\varepsilon > 0$, because $(y, \ell) \mapsto \rho_y(\ell)$ and $(y, \ell) \mapsto \rho_y^{-1}(\ell)$ are continuous in $W^{1,\infty} \times \mathbb{R}$, so $\|w - \tilde{w}\|_{W^{1,\infty}} \rightarrow 0$, $\hat{R} \rightarrow R$ and $\hat{\theta} \rightarrow \theta$ as $\varepsilon \rightarrow 0$. \square

6 Constructing the value function and verification

In this section, we construct a candidate for the value function and verify the variational inequality (3.3) in Lemmas 6.5 and 6.6, relying on results from the previous sections. This will be sufficient to conclude the proof of our main result, Theorem 3.1.

Having defined a candidate boundary via the ODE (5.12), to separate the sell and wait regions \mathcal{S} and \mathcal{W} , we will now construct a solution V of the variational inequality (3.3) that will give the value function of the optimal liquidation problem. As a direct consequence of Lemma 5.5, we get the value along the boundary

$$V_{\text{bdry}}(\theta) := V(y(\theta), \theta) = \Phi(y(\theta)) M_1(y(\theta)). \quad (6.1)$$

Inside the wait region \mathcal{W} , which we assume is to the left of the boundary, we require $V = V^{\mathcal{W}}$ to satisfy $\frac{1}{2}V_{yy} + (\sigma\hat{\rho} - \beta y)V_y = \delta V$. Note that this is actually $\mathcal{G}V = \delta V$, i.e. $V^{\mathcal{W}}$ should be an eigenfunction to the eigenvalue δ of the generator \mathcal{G} of a diffusion as in Theorem 7.3. Since V should be also monotonically increasing, the only possibility is that $V^{\mathcal{W}}(y, \theta) = C(\theta)\Phi(y)$ for some increasing function $C : [0, \infty) \rightarrow [0, \infty)$. Using the boundary condition $V^{\mathcal{W}}(y(\theta), \theta) = V_{\text{bdry}}(\theta)$, with equation (6.1) we then have

$$V^{\mathcal{W}}(y, \theta) := \Phi(y)C(\theta) \quad (6.2)$$

for $y \leq y(\theta)$ and $\theta \geq 0$, where $C(\theta) := M_1(y(\theta))$. In the sell region, we require for $V = V^{\mathcal{S}}$ to satisfy $f = V_y^{\mathcal{S}} + V_\theta^{\mathcal{S}}$. Therefore, we divide \mathcal{S} in two parts:

$$\begin{aligned} \mathcal{S}_1 &:= \{(y, \theta) \in \mathbb{R} \times (0, \infty) \mid y(\theta) < y \leq y_0 + \theta\}, \\ \mathcal{S}_2 &:= \{(y, \theta) \in \mathbb{R} \times (0, \infty) \mid y_0 + \theta < y\}. \end{aligned} \quad (6.3)$$

Let $\Delta := \Delta(y, \theta) \geq 0$ denote the $\|\cdot\|_1$ -distance of a point $(y, \theta) \in \bar{\mathcal{S}}$ to the boundary $\partial\mathcal{S}$ in direction $(-1, -1)$. This means in $\bar{\mathcal{S}}_1$ (but not in \mathcal{S}_2) that

$$y(\theta - \Delta) = y - \Delta. \quad (6.4)$$

In the following let $(y, \theta) \in \bar{\mathcal{S}}_1$ and $(y_b, \theta_b) := (y - \Delta, \theta - \Delta) \in \mathcal{W} \cap \mathcal{S}$. We find $\Delta_y = 1/(1 - y'(y_b)) = 1 - \Delta_\theta$. Inside $\bar{\mathcal{S}}_1$, we need to have

$$V^{\mathcal{S}_1}(y, \theta) := V^{\mathcal{W}}(y - \Delta, \theta - \Delta) + \int_{y-\Delta}^y f(x) dx, \quad (6.5)$$

since $V_y^{\mathcal{S}_1} + V_\theta^{\mathcal{S}_1} = f$ in $\bar{\mathcal{S}}$ and $V^{\mathcal{S}_1}(y(\theta), \theta) = V_{\text{bdry}}(\theta) = V^{\mathcal{W}}(y(\theta), \theta)$. Similarly, in \mathcal{S}_2 ,

$$V^{\mathcal{S}_2}(y, \theta) := \int_{y-\theta}^y f(x) dx. \quad (6.6)$$

To wrap up, the candidate value function is defined by:

$$V = V^{\mathcal{W}} \text{ on } \bar{\mathcal{W}}, \quad V = V^{\mathcal{S}_1} \text{ on } \bar{\mathcal{S}}_1, \quad V = V^{\mathcal{S}_2} \text{ on } \bar{\mathcal{S}}_2. \quad (6.7)$$

Now we verify that V is a classical solution of the variation inequality equation (3.3) with the boundary condition $V(y, 0) = 0$ for all $y \in \mathbb{R}$. That $V(y, 0) = 0$ is clear because $M_1(y_0) = 0$. The rest will be split into several lemmas.

Lemma 6.1 (Smooth pasting). *Let $(y_b, \theta_b) \in \bar{\mathcal{W}} \cap \bar{\mathcal{S}}$. Then*

$$\Phi(y_b)C'(\theta_b) + \Phi'(y_b)C(\theta_b) = f(y_b), \quad (6.8)$$

$$\Phi'(y_b)C'(\theta_b) + \Phi''(y_b)C(\theta_b) = f'(y_b). \quad (6.9)$$

Proof. This follows easily from the fact that $C(\theta_b) = M_1(y_b)$ and $C'(\theta_b) = M_2(y_b)$, see the definition of C and (5.12), together with the definitions of M_1 and M_2 , see (5.11). Note that when $(y_b, \theta_b) = (y_0, 0)$ we take the right derivative of C at 0 and the equalities still hold true. \square

Remark 6.2. It might be interesting to point out that (6.8) and (6.9) are sufficient to derive the boundary between the sell and the wait regions. Indeed, solving (6.8) – (6.9) with respect to $C(\theta_b)$ and $C'(\theta_b)$, it is easy to see that $C(\theta_b) = M_1(y_b)$ and $C'(\theta_b) = M_2(y_b)$. On the other hand, by the chain rule one gets $\theta'(y_b)C'(\theta_b) = M_1'(y_b)$ from where we derive for the boundary $\theta(\cdot)$ in the appropriate range

$$\theta'(y_b) = \frac{M_1'}{M_2}(y_b),$$

which gives the ODE for the boundary in (5.12). To get the initial condition y_0 , note that the boundary condition $V(\cdot, 0) \equiv 0$ gives $C(0) = 0$, i.e. $M_1(y_0) = 0$, exactly as in Lemma 5.3. Hence, one could guess the candidate boundary $y(\cdot)$ if one assumes sufficient smoothness of the function V along the boundary. This is similar to the usual approach in the singular stochastic control literature, cf. [KS86, Section 6]. The reason why we chose the seemingly longer derivation via calculus of variation techniques is the local (one-sided) optimality that we derived in Theorem 5.6 and that will be crucial in our verification of the inequalities of the candidate value function in the sell region, see Lemma 6.6 below. In the special case of $\lambda(\cdot)$ being constant, a more direct approach to verify the variational inequality is suggesting new, yet unproven (to our best knowledge), properties for quotients of parabolic cylinder functions that might be of independent interest, see Remark 6.7.

To prove regularity of V in Lemma 6.4 below, we will use the smooth pasting property. Moreover, a growth condition of V will be needed to justify later why the stochastic integrals in (3.2) are true martingales by application of the following technical

Lemma 6.3. *Let $\Theta_{0-} \geq 0$ be given and $F \in C^{2,1}(\mathbb{R} \times [0, \infty); \mathbb{R})$ be such that there exist constants $C_1, C_2 \geq 0$ with $|F(y, \theta)| \leq C_1 \exp(C_2 y) \vee 1$ for all $(y, \theta) \in \mathbb{R} \times [0, \Theta_{0-}]$. For an admissible strategy $A \in \mathcal{A}(\Theta_{0-})$ let $Y^A =: Y$ denote the impact process defined by (2.6) for $y \in \mathbb{R}$. Then the stochastic integral processes*

$$\int_0^\cdot \bar{S}_u F(Y_u, \Theta_u) dB_u \quad \text{and} \quad \int_0^\cdot \bar{S}_u F(Y_u, \Theta_u) dW_u \quad \text{are true martingales.}$$

Proof. By the exponential growth of F it suffices to check $\mathbb{E}[\int_0^t \bar{S}_u^2 \exp(2C_2 Y_u) du] < \infty$ for every $t \geq 0$. Consider the Ornstein-Uhlenbeck process $dX_t = -\beta X_t dt + dB_t$, with $X_0 = y$. By applying Itô's formula to the processes $e^{\beta t} X_t$ and $e^{\beta t} Y_t$, one gets that

$$e^{\beta t} (Y_t - X_t) = \int_{[0,t]} e^{\beta u} d\Theta_u \quad \text{for } t \geq 0.$$

Since Θ is non-increasing, we conclude that $Y_t \leq X_t$ for all $t \geq 0$. In particular,

$$\begin{aligned} \mathbb{E}\left[\int_0^t \bar{S}_u^2 \exp(2C_2 Y_u) du\right] &\leq \mathbb{E}\left[\int_0^t \bar{S}_u^2 \exp(2C_2 X_u) du\right] \\ &= \int_0^t \mathbb{E}[\bar{S}_u^2 \exp(2C_2 X_u)] du \leq \int_0^t \sqrt{\mathbb{E}[\bar{S}_u^4] \mathbb{E}[\exp(4C_2 X_u)]} du < \infty, \end{aligned}$$

using Cauchy-Schwarz and the fact that X is a Gaussian process. \square

Lemma 6.4. *The function V is $C^{2,1}(\mathbb{R} \times [0, \infty))$. Moreover, for every Θ_{0-} there exist constants C_1, C_2 such that*

$$0 \leq V_y(y, \theta), \quad V(y, \theta) \leq C_1 \exp(C_2 y) \vee 1 \quad \forall (y, \theta) \in \mathbb{R} \times [0, \Theta_{0-}].$$

Proof. In \mathcal{W} , V is already $C^{2,1}$ by construction and the fact that $C(\theta) = M_1(y(\theta))$ is continuously differentiable since $y(\cdot)$ and $M_1(\cdot)$ are so.

For $(y, \theta) \in \mathcal{S}_1$, set $(y_b, \theta_b) := (y - \Delta(y, \theta), \theta - \Delta(y, \theta))$ and $\Delta := \Delta(y, \theta)$. We have by (6.5) for the first and (6.8) for the second equality

$$\begin{aligned} V_y^{\mathcal{S}_1} &= \Phi'(y_b) C(\theta_b) (1 - \Delta_y) + \Phi(y_b) C'(\theta_b) (-\Delta_y) + f(y) - f(y_b) (1 - \Delta_y) \\ &= \Phi'(y - \Delta) C(\theta - \Delta) + f(y) - f(y - \Delta), \end{aligned} \tag{6.10}$$

Since f, Δ, C and Φ are continuously differentiable, V_y will also be so. Hence by (6.9),

$$\begin{aligned} V_{yy}^{\mathcal{S}_1} &= \Phi''(y_b) C(\theta_b) (1 - \Delta_y) + \Phi'(y_b) C'(\theta_b) (-\Delta_y) + f'(y) - f'(y_b) (1 - \Delta_y) \\ &= V_{yy}^{\mathcal{W}}(y_b, \theta_b) + f'(y) - f'(y_b), \end{aligned} \tag{6.11}$$

which is continuous. On the other hand, by (6.5) and (6.9) we have

$$\begin{aligned} V_{\theta}^{\mathcal{S}_1}(y, \theta) &= \Phi'(y_b)C(\theta_b)(-\Delta_{\theta}) + \Phi(y_b)C'(\theta_b)(1 - \Delta_{\theta}) - f(y_b)(-\Delta_{\theta}) \\ &= \Phi(y_b)C'(\theta_b), \end{aligned} \quad (6.12)$$

which is continuous. For $(y, \theta) \in \overline{\mathcal{W}} \cap \overline{\mathcal{S}}$ on the boundary, the left derivative w.r.t. y is

$$\lim_{x \searrow 0} \frac{1}{x} (V(y, \theta) - V(y - x, \theta)) = \Phi(y)C(\theta),$$

while the right derive is again given by (6.10) and is equal to the left derivative since $\Delta(y, \theta) = 0$ in this case. Hence, V is continuously differentiable w.r.t. y on the boundary with derivative $V_y(y, \theta) = \Phi'(y)C(\theta)$. Similarly, the left derivative of V_y on the boundary is $\Phi''(y)C(\theta)$ and equal to the right derivative which is given by (6.11) with $y = y_b$. Similarly, the left derivative of V w.r.t. θ on the boundary is equal to the right derivative (given by (6.12)). Therefore, V is $C^{2,1}$ inside $\overline{\mathcal{W}} \cup \mathcal{S}_1$.

For $(y, \theta) \in \mathcal{S}_2$, we have by (6.6) that $V_y^{\mathcal{S}_2} = f(y) - f(y - \theta)$, $V_{yy}^{\mathcal{S}_2} = f'(y) - f'(y - \theta)$ and $V_{\theta}^{\mathcal{S}_2} = f(y - \theta)$, which are all continuous.

On the boundary between \mathcal{S}_1 and \mathcal{S}_2 , the left derivative of V w.r.t. y is given by (6.10) while the right derivative is $f(y) - f(y_0)$. Since $\theta - \Delta = 0$ in this case and $C(0) = 0$, they are equal and hence V is continuously differentiable w.r.t. y there. Similarly for V_{yy} there. The left derivative of V w.r.t. θ there is given by (6.12) with $(y_b, \theta_b) = (y_0, 0)$. The right derivative w.r.t. θ is $f(y - \theta) = f(y_0)$. They are equal by (6.9) and $C(0) = 0$. Therefore, V is $C^{2,1}$ on $\overline{\mathcal{S}}_1 \cup \mathcal{S}_2$.

It remains to check smoothness on the boundary $\{(y, 0) \mid y \in \mathbb{R}\}$. The derivatives w.r.t. y there are 0. V is continuously differentiable w.r.t. θ in this case because $y(\cdot)$, C , and Δ are continuously differentiable w.r.t. θ also at $\theta = 0$ (we consider the right derivatives there).

To conclude the proof, the growth condition follows as follows. In the wait region, which is contained in $(-\infty, y_0] \times [0, \infty)$, we have $V(y, \theta) = C(\theta)\Phi(y)$ and $V_y(y, \theta) = C(\theta)\Phi'(y)$. Since Φ, Φ' are strictly increasing in y (see (4.9) and the properties after it), V and V_y will be bounded by a constant there. Now, in the sell region we have $f - V_y - V_{\theta} = 0$. However, $V_{\theta} > 0$ because in \mathcal{S}_1 (6.12) holds and $C'(\theta_b) = M_2(y(\theta_b)) > 0$, while in \mathcal{S}_2 we have that $V_{\theta}(y, \theta) = f(y - \theta) > 0$. Similarly, $V_y > 0$ in the sell region. Therefore, $0 < V_y(y, \theta) < f(y) \leq \exp(\lambda_{\infty}y)$ by Assumption C4, and hence also the exponential bound on V . \square

Next we prove that V solves the variational inequality (3.3).

Lemma 6.5. *The function $V^{\mathcal{W}} : \overline{\mathcal{W}} \rightarrow [0, \infty)$ from (6.2) satisfies*

$$\mathcal{L}V^{\mathcal{W}}(y, \theta) = 0 \quad \text{and} \quad f(y) < V_y^{\mathcal{W}}(y, \theta) + V_{\theta}^{\mathcal{W}}(y, \theta) \text{ for } y < y(\theta).$$

Proof. We have $V_y^{\mathcal{W}} = \Phi'(y)M_1(y(\theta))$ and $V_{\theta}^{\mathcal{W}} = \Phi(y)M_1'(y(\theta))y'(\theta) = \Phi(y)M_2(y(\theta))$ by (5.10). Recall that at $y = y(\theta)$ we have by (6.8) the equality $V_y^{\mathcal{W}} + V_{\theta}^{\mathcal{W}} = f(y(\theta))$. Now consider $y < y(\theta)$. By Lemma 5.2, we then have $M_1(y) > M_1(y(\theta))$ giving

$$\left(\frac{f}{\Phi}\right)'(y) > \left(\frac{\Phi'}{\Phi}\right)'(y)M_1(y(\theta)) = \frac{d}{dy} \left(M_1(y(\theta)) \frac{\Phi'(y)}{\Phi(y)} + M_2(y(\theta)) \right).$$

Therefore, $y \mapsto (f - V_y^{\mathcal{W}}(y, \theta) + V_\theta^{\mathcal{W}}(y, \theta))/\Phi(y)$ is increasing in y . Since at $y = y(\theta)$ it equals to 0, we get the claimed inequality. \square

It remains to verify the inequality in the sell region. The proof is the more subtle and that is where Theorem 5.6 plays a crucial role. Recall Assumption 2.2 and note that y_∞ from Lemma 5.3 is unique by condition C3.

Lemma 6.6. *The functions $V^{\mathcal{S}_1}$ and $V^{\mathcal{S}_2}$ satisfy on $\bar{\mathcal{S}}_1$ and \mathcal{S}_2 respectively*

$$\mathcal{L}V^{\mathcal{S}_1} \leq 0, \quad \mathcal{L}V^{\mathcal{S}_2} < 0.$$

Moreover, the inequality inside $\bar{\mathcal{S}}_1$ is strict except on the boundary between the wait region and the sell region ($\bar{\mathcal{W}} \cap \bar{\mathcal{S}}_1$) where we have equality.

Proof. First consider region $\bar{\mathcal{S}}_1$. Recall from Lemma 6.4 (see (6.10) -(6.11)) that

$$\begin{aligned} V_y^{\mathcal{S}_1} &= V_y^{\mathcal{W}}(y - \Delta, \theta - \Delta) + f(y) - f(y - \Delta), \\ V_{yy}^{\mathcal{S}_1} &= V_{yy}^{\mathcal{W}}(y_b, \theta_b) + f'(y) - f'(y_b). \end{aligned}$$

Fix $(y_b, \theta_b) \in \bar{\mathcal{W}} \cap \bar{\mathcal{S}}_1$ and consider the perturbation $\Delta \mapsto (y, \theta) = (y_b + \Delta, \theta_b + \Delta)$. Set

$$h(\Delta) := \mathcal{L}V^{\mathcal{S}_1}(y_b + \Delta, \theta_b + \Delta) \quad (6.13)$$

$$\begin{aligned} &= \frac{1}{2}V_{yy}^{\mathcal{W}}(y_b, \theta_b) - \frac{1}{2}f'(y_b) + \sigma\hat{\rho}V_y^{\mathcal{W}}(y_b, \theta_b) - \sigma\hat{\rho}f(y_b) - \delta V^{\mathcal{W}}(y_b, \theta_b) \\ &\quad + \frac{1}{2}f'(y) - \beta y V_y^{\mathcal{W}}(y_b, \theta_b) + \beta y f(y_b) + (\sigma\hat{\rho} - \beta y)f(y) - \delta \int_{y_b}^y f(x) dx. \end{aligned} \quad (6.14)$$

We have $h(0) = 0$ by Lemma 6.5 and to show $h(\Delta) < 0$ for $\Delta > 0$, it suffices to prove $h'(\Delta) < 0$ for all $\Delta > 0$. We have for all $\Delta \geq 0$ at $y = y_b + \Delta$ that

$$h'(\Delta) = \beta(f(y_b) - V_y^{\mathcal{W}}(y_b, \theta_b)) + f(y) \underbrace{\left(\frac{1}{2} \frac{f''(y)}{f(y)} - (\beta + \delta) + (\sigma\hat{\rho} - \beta y) \frac{f'(y)}{f(y)} \right)}_{=k(y)}, \quad (6.15)$$

where at $\Delta = 0$ we consider the right derivative $h'(0+)$. Now we show that $k(y) < 0$ for all $y \geq y_\infty$. Recall that Φ is a solution of the ODE $0 = \frac{1}{2}\Phi''(x) + (\sigma\hat{\rho} - \beta x)\Phi'(x) - \delta\Phi(x)$. Differentiating w.r.t. x and dividing by $\Phi'(x)$ yields

$$0 = \frac{1}{2} \left(\frac{\Phi''(x)}{\Phi'(x)} \right)' + \frac{1}{2} \frac{\Phi''(x)^2}{\Phi'(x)^2} - (\beta + \delta) + (\sigma\hat{\rho} - \beta x) \frac{\Phi''(x)}{\Phi'(x)} \quad (6.16)$$

So at the left end y_∞ of our boundary, we have

$$\begin{aligned} k(y_\infty) &= \frac{1}{2} \left(\frac{f'}{f} \right)'(y_\infty) + \frac{1}{2} \frac{\Phi''(y_\infty)^2}{\Phi'(y_\infty)^2} - (\beta + \delta) + (\sigma\hat{\rho} - \beta y_\infty) \frac{\Phi''(y_\infty)}{\Phi'(y_\infty)} \\ &= \frac{1}{2} \left(\frac{f'}{f} \right)'(y_\infty) - \frac{1}{2} \left(\frac{\Phi''}{\Phi'} \right)'(y_\infty) < 0 \end{aligned} \quad (6.17)$$

by Assumption C3. With Assumption C6 we get $k(y) < 0$ for every $y \geq y_\infty$.

In particular, $k(y_b + \Delta) < 0$ for all $\Delta \geq 0$. Since f is positive and increasing, the product $\Delta \mapsto (fk)(y_b + \Delta)$ is decreasing. Therefore, proving $h'(0+) \leq 0$ is sufficient to show the inequality in \mathcal{S}_1 . To stress the dependence of h on the point $(y_b, \theta_b) = (y(\theta_b), \theta_b)$, we also write $h(\Delta) = h_{\theta_b}(\Delta)$. Note that $h_\theta(\Delta)$ is continuous in θ and Δ on $[0, \infty) \times [0, \infty)$.

Assume $h'_{\theta_b}(0+) > 0$ at some boundary point (y_b, θ_b) with $\theta_b > 0$. By continuity of h' on θ and Δ there exists some $\varepsilon > 0$ such that $\mathcal{L}V^{\mathcal{S}_1} > 0$ on $U := \bar{\mathcal{S}}_1 \cap B_\varepsilon(y_b, \theta_b)$. This will lead to a contradiction to the fact that the candidate boundary is a (one-sided) strict local maximizer of our stochastic optimization problem with strategies described by the local times of reflected diffusions, see Theorem 5.6.

Indeed, fix $\Theta_0 > \theta_b + \varepsilon$ and consider a perturbation $\tilde{y}(\cdot) \in C^1$ of the boundary $y(\cdot)$ which satisfies the conditions of Theorem 5.6 as well as $y(\theta) < \tilde{y}(\theta) \leq y_0$ in $(\tilde{y}(\theta), \theta) \in U$ and such that \tilde{y} and y coincide outside of U . For the corresponding reflection strategies $\tilde{A} := A^{\text{refl}}(\tilde{y})$ and $A := A^{\text{refl}}(y)$ denote by $\tilde{\Theta}_t := \Theta_0 - \tilde{A}_t$ and $\Theta_t := \Theta_0 - A_t$ their asset position processes. The liquidation times of \tilde{A} and A are $\tilde{\tau} := \inf\{t \geq 0 \mid \tilde{A}_t = \Theta_0\}$ and $\tau := \inf\{t \geq 0 \mid A_t = \Theta_0\}$, respectively. By Theorem 7.3, we have $T := \tilde{\tau} \vee \tau < \infty$ a.s. Fix initial impact $Y_{0-}^{\tilde{A}} = Y_{0-}^A = y(\Theta_0)$. To compare the strategies A and \tilde{A} , consider the processes $G(y(\Theta_0), A)$ and $G(y(\Theta_0), \tilde{A})$ for our candidate value function (which is $C^{2,1}$ by Lemma 6.4). Since $V(\cdot, 0) = 0$, it holds $L_T(\tilde{A}) = G_T(\tilde{A})$ and $L_T(A) = G_T(A)$. However, since $(Y^{\tilde{A}}, \tilde{\Theta})$ spends a positive amount of time in the $\{\mathcal{L}V > 0\}$ region until time T and always remains in the region $\{\mathcal{L}V \geq 0\}$, the perturbed strategy \tilde{A} generates larger proceeds (in expectation) than A .

Indeed, by (3.2) for $G(\tilde{A})$ and $G(A)$, using monotone convergence and Lemma 6.3 for the stochastic integrals (noting growth condition from Lemma 6.4) we get

$$\mathbb{E}[L_\infty(\tilde{A}) - L_\infty(A)] = \lim_{n \rightarrow \infty} \mathbb{E}[G_{n \wedge T}(\tilde{A}) - G_{n \wedge T}(A)] \quad (6.18)$$

$$= \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_0^{n \wedge T} \dots dW_t - \int_0^{n \wedge T} \dots dB_t + \int_0^{n \wedge T} \mathcal{L}V(Y_t^{\tilde{A}}, \tilde{\Theta}_t) dt\right] \quad (6.19)$$

$$= \mathbb{E}\left[\int_0^T \mathcal{L}V(Y_t^{\tilde{A}}, \tilde{\Theta}_t) dt\right] > 0. \quad (6.20)$$

This contradicts Theorem 5.6, so $h'(0+) \leq 0$ and hence the inequality in \mathcal{S}_1 must hold. It remains to consider $(y, \theta) \in \bar{\mathcal{S}}_2$. Note that $V_y^{\mathcal{S}_2} = f(y) - f(y - \theta)$ and $V_{yy}^{\mathcal{S}_2} = f'(y) - f'(y - \theta)$. Fix $y - \theta =: a \geq y_0$. Consider $\mathcal{L}V^{\mathcal{S}_2}$ as function of θ . We have

$$\mathcal{L}V^{\mathcal{S}_2}(y, \theta) = \frac{1}{2}(f'(a + \theta) - f'(a)) + (\sigma\hat{\rho} - \beta(a + \theta))(f(a + \theta) - f(a)) - \delta \int_a^{a + \theta} f(x) dx.$$

Differentiating the right-hand side w.r.t. θ we get the expression $f(a + \theta)k(a + \theta)$, which is again decreasing in θ because $a \geq y_0$. Since at $\theta = 0$ we have $\mathcal{L}V^{\mathcal{S}_2}(y, \theta) = 0$ we deduce the desired inequality. \square

Remark 6.7 (A conjecture on properties of special functions). In the particular case when $\lambda = f'/f$ is a constant function, a more direct approach based on direct

calculations leads to a conjecture on a property for quotients of Hermite functions. More precisely, to prove $h'(0+) \leq 0$ in this case it turns out to be sufficient to verify that the map $y_b \mapsto h'(0)$ is monotone in $[y_\infty, y_0]$ because at y_∞ and y_0 one can check that $h'(0+) < 0$. The monotonicity in y_b would then follow from the following conjectured property of the Hermite functions:

$$\text{For every } \nu < 0, \text{ the function } x \mapsto \frac{(H_{\nu-1}(x))^2}{H_\nu(x)H_{\nu-2}(x)} \text{ is decreasing.}$$

Numerical computations indicate the validity of the this property but, to our best knowledge, it is not yet proven and may be of independent interest. Note that such quotients of special functions are related to so called Turan-type inequalities, cf. [BI13].

Now we have all the ingredients in place to complete the

Proof of Theorem 3.1. The function V constructed in (6.7) is a classical solution of the variational inequality (3.3) because of Lemmas 6.4, 6.5 and 6.6. Thus, for each admissible strategy A the process $G(y; A)$ from (3.1) is a supermartingale with $G_0(y; A) \leq G_{0-}(y; A)$: the growth condition on V_y and V from Lemma 6.4 guarantee that the stochastic integral processes in (3.2) are true martingales by an application of Lemma 6.3, while the variational inequality give the supermartingale property on $[0-, \infty)$. Moreover, for the described strategy A^* , whose existence and uniqueness on $\llbracket 0, \tau \rrbracket$ follow from Theorem 7.3, the process $G(y; A^*)$ is a true martingale with $G_0(y; A^*) = G_{0-}(y; A^*)$ by our construction of V and the validity of the variational inequality in the respective regions. Any other strategy will be suboptimal because the respective inequalities are strict in the sell and wait region, i.e., for any other strategy the process G will be a strict supermartingale.

The Laplace transform formula will be derived in Theorem 7.3 for a y -reflected strategy when the state process starts on the boundary. If the state process starts in the wait region, note that the behavior of the process until it hits the boundary for the first time is independent from future excursions from the boundary; this explains the multiplicative factor in (3.4), cf. equation (7.8). \square

7 SDEs with reflection at an elastic boundary

Section 4 requires for specific reflected diffusion processes the Laplace transform of inverse local times on boundaries which depend on local time. As the analysis appears interesting in itself and we do not see benefits from the particular OU structure for our arguments, we present our results here in a self-contained way for a general setup. The classical Skorokhod problem is that of reflecting a path at a boundary. It is the standard tool for constructing solutions to SDEs with reflecting boundary conditions, where the boundary remains constant over time. The most basic example is a Brownian motion with values in $[0, \infty)$ and reflection at zero, solved in [Sko61]. Starting with [Tan79], well known generalizations concern diffusions in multiple dimensions with normal or oblique reflection at the boundary of some given (time-invariant) domain in

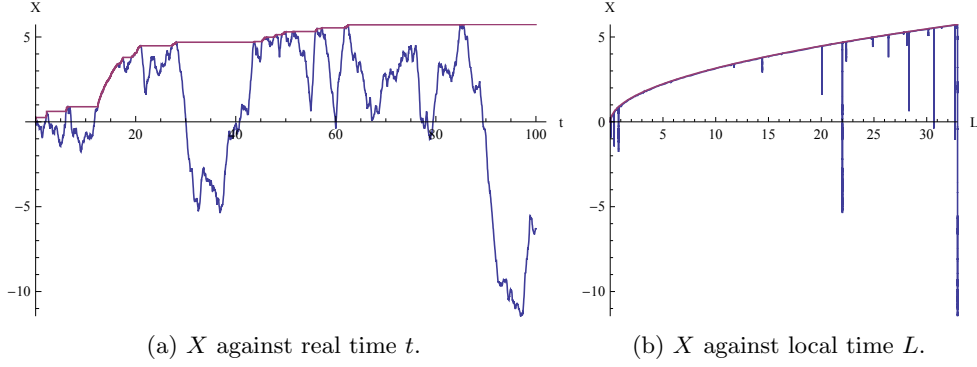


Figure 1: Brownian Motion X (blue) reflected at the elastic boundary \sqrt{L} (purple) against different clocks, where L is the local time of X at that boundary.

the Euclidian space of certain smoothness or other kinds of regularity, cf. [LS84, DI93]. Other generalizations admit for an a-priori given but time-dependent boundary, cf. [EKK91, NÖ10]. In the present section we restrict ourselves to a one-dimensional setting but with a local-time-dependent boundary, as we need it in Section 4, with the interpretation that the boundary interacts with the diffusion, cf. Fig. 1a. Taking local time L at the boundary as a second coordinate, this can be seen as a special case of a degenerate diffusion (X, L) in \mathbb{R}^2 with oblique reflection at a smooth boundary, see Fig. 1b. The main result (Theorem 7.3) is an explicit construction of (X, L) through approximations of reflections by small jumps and the explicit formula (7.12) for the Laplace transform of the inverse local time.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with a one-dimensional (\mathcal{F}_t) -Brownian motion W and a filtration (\mathcal{F}_t) satisfying the usual conditions of right-continuity and completeness. For the rest of this section, we assume $\sigma : \mathbb{R} \rightarrow (0, \infty)$ and $b : \mathbb{R} \rightarrow \mathbb{R}$ to be Lipschitz-continuous functions such that the continuous \mathbb{R} -valued (b, σ) -diffusion with generator $\mathcal{G} := \frac{1}{2}\sigma(x)^2 \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$ is regular and recurrent. Moreover, let $g \in C^1([0, \infty))$ be non-decreasing.

Let X be a (b, σ) -diffusion with reflection at an elastic boundary. This means the boundary moves back whenever it is hit by the diffusion. More precisely,

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t - dL_t, \quad X_0 = g(0), \quad (7.1)$$

with continuous non-decreasing (local time) process L of X at the (time-dependent) boundary $g(L)$ such that

$$dL_t = \mathbf{1}_{\{X_t = g(L_t)\}} dL_t, \quad L_0 = 0, \quad \text{with } X_t \leq g(L_t) \text{ for all } t \geq 0. \quad (7.2)$$

We are particularly interested in the inverse local time

$$\tau_\ell := \inf\{t > 0 \mid L_t > \ell\}. \quad (7.3)$$

Let H^y denote the first hitting time of a point y by some (b, σ) -diffusion. We write $H^{x \rightarrow z}$ for the hitting time when the diffusion starts in x . Note that $\mathbb{P}[H^{x \rightarrow y} < \infty] = 1$ for all x, y by our assumption on the diffusion being regular and recurrent.

Remark 7.1. Note that $\{t \geq 0 \mid X_t = g(L_t)\}$ is a.s. of Lebesgue measure zero by [RY99, ex. VI.1.16]. For a constant boundary $g(\ell) \equiv a$, Tanaka's formula for symmetric local times [RY99, ex. VI.1.25] hence shows that the process L , that we obtain as a solution to the SDE with reflection (7.1)-(7.2), is the symmetric local time of the continuous semimartingale X at given level $a \in \mathbb{R}$, i.e. $L_t = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(a-\varepsilon, a+\varepsilon)}(X_s) d\langle X, X \rangle_s$.

7.1 Approximation by ε -reflections

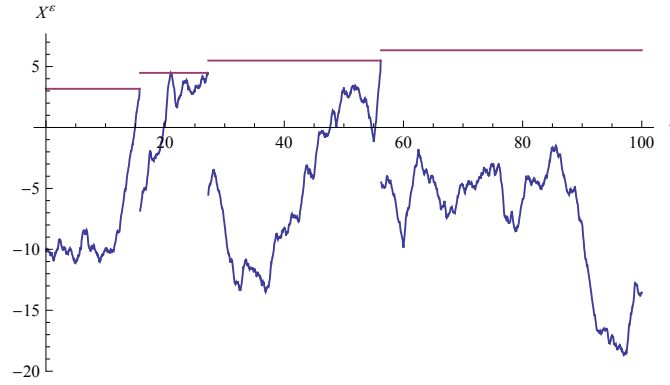


Figure 2: Approximation X^ε for $\varepsilon = 10$

We construct solutions to (7.1)–(7.2) and derive explicit representation (7.12) of the Laplace transform of the inverse local time at boundary g by approximating reflection by jumps in the system of SDEs below.

$$dX_t^\varepsilon = b(X_t^\varepsilon) dt + \sigma(X_t^\varepsilon) dW_t - dL_t^\varepsilon, \quad X_{0-}^\varepsilon := g(0), \quad (7.4)$$

$$L_t^\varepsilon := \sum_{0 \leq s \leq t} \Delta L_s^\varepsilon \quad \text{with } \Delta L_t^\varepsilon := \begin{cases} \varepsilon & \text{if } X_{t-}^\varepsilon = g(L_{t-}^\varepsilon), \\ 0 & \text{otherwise,} \end{cases} \quad L_{0-}^\varepsilon := 0, \quad (7.5)$$

$$\tau_\ell^\varepsilon := \inf\{t > 0 \mid L_t^\varepsilon > \ell\} \quad \text{for } \ell \geq 0. \quad (7.6)$$

As soon as process X^ε hits the boundary, it is reflected by a jump of fixed size $\varepsilon > 0$. We will speak of L^ε as discrete local time, as it is approximating L in the sense of Theorem 7.3. Since the target reflected diffusion X starts at the boundary g , we now have $X_0^\varepsilon = g(0) - \varepsilon$ after an initial jump $\Delta L_0^\varepsilon = \varepsilon$ away from $X_{0-}^\varepsilon := g(0)$.

Lemma 7.2. *Let g , b and σ satisfy the assumptions of this section (cf. p. 25). For every $\varepsilon > 0$, the above SDE has a unique (up to indistinguishability) strong solution $(X_t^\varepsilon, L_t^\varepsilon)_{t \geq 0}$ on $[0, \infty)$, and uniqueness in law holds (for solutions living on different filtered probability spaces, with different W).*

Proof. Indeed, one can argue by results [RW87, V.9–11, V.17] for classical diffusion SDEs with Lipschitz coefficients (b, σ) by inductive construction on $\llbracket 0, \tau_n \rrbracket$ where $\tau_n := \inf\{t > \tau_{n-1} \mid X_{t-}^\varepsilon = g(n\varepsilon)\} = \tau_{\varepsilon n}^\varepsilon$ for $n \geq 1$, with $\tau_0 := 0$. Clearly L_t^ε equals $L_{\tau_{n-1}}^\varepsilon$ for $t \in \llbracket \tau_{n-1}, \tau_n \rrbracket$ and $L_{\tau_n}^\varepsilon = L_{\tau_{n-1}}^\varepsilon + \varepsilon$, while $X_u^\varepsilon = F(X_{\tau_{n-1}}^\varepsilon, (W_{\tau_{n-1}+s})_{s \geq 0})_{u-\tau_{n-1}}$ on $\llbracket \tau_{n-1}, \tau_n \rrbracket$ holds for a suitable functional representation F of strong solutions to (b, σ) -diffusions [RW87, Theorem V.10.4]. Such construction extends to $\llbracket 0, \tau_\infty \rrbracket$ for the monotone limit $\tau_\infty := \lim_n \tau_n$.

It suffices to argue that $\tau_\infty = \infty$ (a.s.). To this end, let us consider

$$\tau'_n := \inf\{t > \tau_{n-1} \mid X_{t-}^\varepsilon = g((n-1)\varepsilon)\}, \quad n \geq 1,$$

so that $\tau_{n-1} < \tau'_n \leq \tau_n$ and $X_{\tau'_n}^\varepsilon = g((n-1)\varepsilon) = X_{(\tau_{n-1})-}^\varepsilon$. Considering the time change $\varphi_t := \int_0^t \sum_{n=1}^\infty 1_{\llbracket \tau'_n, \tau_n \rrbracket} du$ with right-continuous inverse $s_t := \inf\{u \mid \varphi_u > t\}$, one has (using [RW87, IV.30.10]) that $X'_t := X_{s_t}^\varepsilon$, $t \geq 0$, is the solution to the diffusion SDE $dX'_t = b(X'_t)dt + \sigma(X'_t)dW'_t$, $X'_0 = g(0)$, on $\llbracket 0, \varphi_\infty \rrbracket$ for $\varphi_\infty := \sup_t \varphi_t$, with respect to the driving Brownian motion $W'_t = \int_0^{s_t} \sum_{n=1}^\infty 1_{\llbracket \tau'_n, \tau_n \rrbracket} dW_u$.

To show that $\{\tau_\infty < \infty\}$ is a nullset, let $g_\infty := \lim_n g(n\varepsilon) \in \mathbb{R} \cup \{\infty\}$. If $g_\infty = \infty$, we get from $X'_{T_n} = X_{\tau_n}^\varepsilon \rightarrow \infty$ for $T_n := \sum_{i=1}^n (\tau_i - \tau'_i) \leq \tau_n$ and $T_\infty = \lim_n T_n \leq \tau_\infty$ that $\sup_{t < T_\infty} X'_t = \infty$. This implies that $P[\tau_\infty < \infty] = 0$, since the Lipschitz diffusion SDE for X' is pathwise exact, i.p. its continuous solution X' almost surely remains bounded on any finite time interval $\llbracket 0, T \rrbracket$ and $T_\infty \leq \tau_\infty$.

In the case $g_\infty < \infty$, one can find $x, y \in \mathbb{R}$ with $g_\infty - \varepsilon < x < y < g_\infty$. The durations $\tau_n^y - \tau_n^x$, $n \in \mathbb{N}$, for upcrossings of the interval $[x, y]$ between $\tau_n^x := \inf\{t \geq \tau_{n-1} \mid X_t^\varepsilon = x\}$ and $\tau_n^y := \inf\{t \geq \tau_{n-1} \mid X_t^\varepsilon = y\}$ are i.i.d., by the strong Markov property of the time-homogeneous diffusion. By the law of large numbers, $\frac{1}{n} \sum_{i=1}^n \exp(-\lambda(\tau_i^y - \tau_i^x))$ converges almost surely for $n \rightarrow \infty$ to the Laplace transform $\mathbb{E}_x[\exp(-\lambda H^y)]$, $\lambda \geq 0$, of the hitting time H^y for y by the (b, σ) -diffusion process (started at x). This expectation is strictly less than 1 for $\lambda > 0$, as $H^y > 0$ P_x -a.s. for $y > x$, whereas the limit on the sequence equals 1 on $\{\tau_\infty < \infty\}$, where $\lim_i (\tau_i^y - \tau_i^x) = 0$. Hence $P[\tau_\infty < \infty] = 0$. \square

By (7.4) – (7.6), we have $\tau_0^\varepsilon = \tau_{0-}^\varepsilon = 0$ and $\tau_\ell^\varepsilon = \tau_{(k-1)\varepsilon}^\varepsilon$ for $\ell \in [(k-1)\varepsilon, k\varepsilon]$ with $k \in \mathbb{N}$, and $\tau_{k\varepsilon}^\varepsilon$ is the k -th jump time of X^ε and L^ε within period $(0, \infty)$. For $\ell = k\varepsilon$, the approximating process X^ε is a continuous (b, σ) -diffusion on stochastic intervals $\llbracket \tau_{\ell-}^\varepsilon, \tau_\ell^\varepsilon \rrbracket$, and $X_{\tau_\ell^\varepsilon}^\varepsilon = X_{\tau_{\ell-}^\varepsilon}^\varepsilon - \varepsilon = g(L_{\tau_{\ell-}^\varepsilon}^\varepsilon) - \varepsilon = g(\ell - \varepsilon) - \varepsilon$. For such $\ell = k\varepsilon$, we shall call $\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon$ the length of the $(k$ -th) excursion of X^ε away from the boundary; Note that this excursion length is independent of $\mathcal{F}_{\tau_{\ell-}^\varepsilon}^\varepsilon$ and its (conditional) distribution is

$$\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon \sim H^{g(\ell)} \quad \text{under } \mathbb{P}_{g(\ell-\varepsilon)-\varepsilon}, \quad (7.7)$$

what is also denoted as $\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon \stackrel{d}{=} H^{g(\ell-\varepsilon)-\varepsilon} \rightarrow g(\ell)$. The Laplace transform of first hitting times H^z of a (b, σ) -diffusion starting in x is well-known, see e.g. [RW87, V.50]. For states $x, z \in \mathbb{R}$ and $\lambda > 0$,

$$\mathbb{E}[e^{-\lambda H^{x \rightarrow z}}] \equiv \mathbb{E}_x[e^{-\lambda H^z}] = \begin{cases} \Phi_{\lambda,-}(x)/\Phi_{\lambda,-}(z) & \text{if } x < z, \\ \Phi_{\lambda,+}(x)/\Phi_{\lambda,+}(z) & \text{if } x > z, \end{cases} \quad (7.8)$$

where functions $\Phi_{\lambda,\pm}$ are uniquely determined up to a constant factor as the increasing ($\Phi_{\lambda,-}$) respectively decreasing ($\Phi_{\lambda,+}$) positive solutions Φ of the differential equation

$$\mathcal{G}\Phi = \lambda\Phi$$

with generator $\mathcal{G} = \frac{1}{2}\sigma(x)^2 \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$ of the (b, σ) -diffusion. Since we assume the boundary function g to be non-decreasing, only $\Phi_{\lambda,-}$ is of interest for our purpose.

Due to independence of Brownian increments over disjoint time intervals, the Laplace transform of the inverse local time can be calculated from a sum of (independent) excursion lengths at (discrete) local times $\ell_n := \varepsilon n$ as

$$\begin{aligned} \mathbb{E}[\exp(-\lambda\tau_\ell^\varepsilon)] &= \mathbb{E}\left[\exp\left(-\lambda \sum_{n=1}^{\lfloor \ell/\varepsilon \rfloor} (\tau_{\ell_n}^\varepsilon - \tau_{\ell_{n-}}^\varepsilon)\right)\right] = \prod_{n=1}^{\lfloor \ell/\varepsilon \rfloor} \mathbb{E}\left[\exp\left(-\lambda(\tau_{\ell_n}^\varepsilon - \tau_{\ell_{n-}}^\varepsilon)\right)\right] \\ &= \prod_{n=1}^{\lfloor \ell/\varepsilon \rfloor} \mathbb{E}_{g(\ell_n - \varepsilon) - \varepsilon}[\exp(-\lambda H^{g(\ell_n)})] = \prod_{n=1}^{\lfloor \ell/\varepsilon \rfloor} \frac{\Phi_{\lambda,-}(g(\ell_n - \varepsilon) - \varepsilon)}{\Phi_{\lambda,-}(g(\ell_n))} \\ &= \exp\left(\sum_{n=1}^{\lfloor \ell/\varepsilon \rfloor} \log\left(\frac{\Phi_{\lambda,-}(g(\ell_n - \varepsilon) - \varepsilon)}{\Phi_{\lambda,-}(g(\ell_n))}\right)\right), \end{aligned} \quad (7.9)$$

for $\ell \geq 0$ and $\lambda > 0$. With $h_n(\xi) := \Phi_{\lambda,-}(g(\ell_n - \xi) - \xi)$, each summand in (7.9) equals

$$\begin{aligned} \log h_n(\varepsilon) - \log h_n(0) &= \int_0^\varepsilon \frac{h'_n(\xi)}{h_n(\xi)} d\xi \\ &= - \int_0^\varepsilon (g'(\ell_n - \xi) + 1) \frac{\Phi'_{\lambda,-}(g(\ell_n - \xi) - \xi)}{\Phi_{\lambda,-}(g(\ell_n - \xi) - \xi)} d\xi \\ &= - \int_{\ell_{n-1}}^{\ell_n} (g'(a) + 1) \frac{\Phi'_{\lambda,-}(g(a) + a - \ell_n)}{\Phi_{\lambda,-}(g(a) + a - \ell_n)} da. \end{aligned} \quad (7.10)$$

Therefore, we obtain

$$\mathbb{E}[\exp(-\lambda\tau_\ell^\varepsilon)] = \exp\left(- \int_0^{\varepsilon \lfloor \ell/\varepsilon \rfloor} (g'(a) + 1) \frac{\Phi'_{\lambda,-}(g(a) + a - \varepsilon \lceil a/\varepsilon \rceil)}{\Phi_{\lambda,-}(g(a) + a - \varepsilon \lceil a/\varepsilon \rceil)} da\right). \quad (7.11)$$

This suggests the following

Theorem 7.3. *Under the assumptions for this section (cf. p.25) on the diffusion coefficients b, σ and on the boundary function g , there exists a pair (X, L) of continuous adapted processes with non-decreasing L , that is (on the given filtered probability space) the unique strong solution to the reflected SDE (7.1) – (7.2) on time interval $[0, \infty)$. The pair $(X_t, L_t)_{t \geq 0}$ is the weak limit (i.e. in distribution) of $(X_t^\varepsilon, L_t^\varepsilon)_{t \geq 0}$ from (7.4) – (7.5) for $\varepsilon \rightarrow 0$. The inverse local time $\tau_\ell := \inf\{t > 0 \mid L_t > \ell\}$ has the Laplace transform*

$$\mathbb{E}[e^{-\lambda\tau_\ell}] = \exp\left(- \int_0^\ell (g'(a) + 1) \frac{\Phi'_{\lambda,-}(g(a))}{\Phi_{\lambda,-}(g(a))} da\right) \quad \text{for } \lambda > 0, \ell \geq 0, \quad (7.12)$$

where $\Phi_{\lambda,-}$ is the (up to a constant factor) unique positive increasing solution of the differential equation $\mathcal{G}\Phi = \lambda\Phi$, for \mathcal{G} denoting the generator of the (b, σ) -diffusion.

Proof. Existence and uniqueness of (X, L) is shown in Lemma 7.13 below. Using dominated convergence for the right-hand side of equation (7.11), we find

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[e^{-\lambda \tau_\ell^\varepsilon}] = \exp\left(-\int_0^\ell (g'(a) + 1) \frac{\Phi'_{\lambda, -}(g(a))}{\Phi_{\lambda, -}(g(a))} da\right).$$

For the left-hand side, it suffices to prove weak convergence $\tau_\ell^\varepsilon \Rightarrow \tau_\ell$ as $\varepsilon \rightarrow 0$ for all $\ell \geq 0$. This is done in Corollary 7.14 below. \square

Remark 7.4. One can view the process (X, L) as a two-dimensional diffusion reflected at the boundary g with constant oblique direction of reflection $(-1, +1)$. While global existence of (X, L) would not immediately follow by general results on SDEs with oblique reflection in domains [DI93], where boundedness of the domain is assumed and used for certain arguments, note that we get it here as a byproduct from the fact that the approximations $(X^\varepsilon, L^\varepsilon)$ are defined globally, cf. Lemma 7.2.

7.2 Tightness and weak convergence

In order to show weak convergence of $(\tau_\ell^\varepsilon)_\varepsilon$, we will prove that the pair of càdlàg processes $(X^\varepsilon, L^\varepsilon)$ forms a tight sequence in $\varepsilon \rightarrow 0$, and utilizing results on weak convergence of SDEs by [KP96], we will show that any limit point (when $\varepsilon \rightarrow 0$) satisfies (7.1) and (7.2). Uniqueness in law for solutions of (7.1) – (7.2) will then allow us to conclude Theorem 7.3.

Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence with $\varepsilon_n \rightarrow 0$ and consider the sequence $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$. To show tightness, we will apply the following (sufficient) tightness criterion due to Aldous.

Proposition 7.5 ([Bil99, Theorem 16.10]). *Let $(E, |\cdot|)$ be a Banach space. If a sequence $(Y^n)_{n \in \mathbb{N}}$ of adapted, E -valued càdlàg processes satisfies the following two conditions, then it is tight.*

- (a) *The sequences $(J_T(Y^n))_n$ and $(Y_0^n)_n$ are tight (in \mathbb{R} and E , respectively) for every $T \in (0, \infty)$, where J_T denotes the largest jump until time T , i.e.*

$$J_T(Y^n) := \sup_{0 < t \leq T} |Y_t^n - Y_{t-}^n|.$$

- (b) *For every $T \in (0, \infty)$ and $\varepsilon_0, \eta > 0$ there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, all (discrete) Y^n -stopping times $\hat{\tau} \leq T$ and all $\delta \in (0, \delta_0]$ we have*

$$\mathbb{P}[|Y_{\hat{\tau}+\delta}^n - Y_{\hat{\tau}}^n| \geq \varepsilon_0] \leq \eta.$$

To get tightness one needs to control both jump size and, regarding $(L_n^\varepsilon)_n$, the frequency of jumps simultaneously. As we are looking at processes with jumps of size $\pm \varepsilon_n \rightarrow 0$, so only the latter is not yet clear. To this end, the next lemma provides a technical bound on $X^{\varepsilon_n}, L^{\varepsilon_n}$, while a second lemma constricts the probability that X^{ε_n} (respectively L^{ε_n}) performs a number of N_n jumps in a time interval of fixed length.

Lemma 7.6 (Upper bound). *Fix a time horizon $T \in (0, \infty)$ and $\eta \in (0, 1]$. Then there exists a constant $M \in \mathbb{R}$ such that $\mathbb{P}[\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M] \leq \eta$, with the domain of definition for the function g being extended by $g(-x) := g(0)$ for $-x < 0$.*

Proof. Consider a continuous (b, σ) -diffusion Y that starts at time $t = 0$ at $g(0)$. For $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots$, let $\ell(n, k) := k\varepsilon_n$. By induction over k , using comparison for diffusion SDEs, cf. [KS91, Theorem 5.2.18], one obtains that (a.s.) $X_t^{\varepsilon_n} \leq Y_t$ for $t \in [0, \tau_{\ell(n, k)}^{\varepsilon_n})$ for all $k \geq 1$, and hence $X^{\varepsilon_n} \leq Y$ on $[0, \infty)$ (a.s.) because $\lim_{k \rightarrow \infty} \tau_{\ell(n, k)}^{\varepsilon_n} = \infty$ for any n by Lemma 7.2. This means that on the event $\{\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M\}$ we have $\sup_{t \in [0, T]} Y_t \geq M$, and hence $H^{g(0) \rightarrow M} \leq T$. Thus

$$\mathbb{P}[\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M] \leq \mathbb{P}[H^{g(0) \rightarrow M} \leq T].$$

Now the claim follows since $\lim_{M \rightarrow \infty} \mathbb{P}[H^{g(0) \rightarrow M} \leq T] = 0$. \square

Lemma 7.7 (Frequency of jumps). *Fix $T \in (0, \infty)$, $\varepsilon_0, \eta > 0$, and set $N_n := \lceil \varepsilon_0 / \varepsilon_n \rceil$. Then there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every bounded stopping time $\hat{\tau} \leq T$ we have $\mathbb{P}[J_{\hat{\tau}, \delta}^{\varepsilon_n} \geq N_n] \leq \eta$ for all $n \geq n_0$, where $J_{\hat{\tau}, \delta}^{\varepsilon_n} := \inf\{k \mid L_{\hat{\tau}}^{\varepsilon_n} + k\varepsilon_n \geq L_{\hat{\tau} + \delta}^{\varepsilon_n}\}$ is the number of jumps of X^{ε_n} , respectively L^{ε_n} , in time $[\hat{\tau}, \hat{\tau} + \delta]$*

Proof. We will first find an estimate for the jump count probability for arbitrary but fixed $\delta > 0$, $n \in \mathbb{N}$, $N_n \in \mathbb{N}$ and $\hat{\tau} \leq T$. Only in part 2) of the proof we will consider $(N_n)_{n \in \mathbb{N}}$ as stated, to study the limit $n \rightarrow \infty$. More precisely, we will show in part 1) that, given $\mathcal{F}_{\hat{\tau}}$, for every $\lambda > 0$ there exist $k_{n, \lambda} \in \{0, 1, \dots, N_n - 1\}$ s.t. for $x_n := g(L_{\hat{\tau}}^{\varepsilon_n} + \varepsilon_n k_{n, \lambda})$,

$$\mathbb{P}[J_{\hat{\tau}, \delta}^{\varepsilon_n} \geq N_n \mid \mathcal{F}_{\hat{\tau}}] \leq e^{\lambda \delta} \left(\frac{\Phi_{\lambda, -}(x_n - \varepsilon_n)}{\Phi_{\lambda, -}(x_n)} \right)^{N_n - 1}. \quad (7.13)$$

1) In this part, consider arbitrary but fixed $\delta > 0$, $n \in \mathbb{N}$, $N_n \in \mathbb{N}$ and $\hat{\tau} \leq T$. We enumerate the jumps and estimate the sum of excursion lengths by δ . Let $\ell_k := L_{\hat{\tau}}^{\varepsilon_n} + k\varepsilon_n$ be the (discrete) local time of the k -th jump after time $\hat{\tau}$. If X^{ε_n} has at least N_n jumps in the interval $[\hat{\tau}, \hat{\tau} + \delta]$, it is doing at least $N_n - 1$ complete excursions (cf. (7.7)), so that, noting that $\tau_{L_t^{\varepsilon_n} - \varepsilon_n}^{\varepsilon_n} \leq t < \tau_{L_t^{\varepsilon_n}}^{\varepsilon_n}$ (for all $t \geq 0$) and $\ell_{N_n - 1} + \varepsilon_n \leq L_{\hat{\tau} + \delta}^{\varepsilon_n}$, we have

$$\delta = (\hat{\tau} + \delta) - \hat{\tau} \geq \tau_{L_{\hat{\tau} + \delta}^{\varepsilon_n} - \varepsilon_n}^{\varepsilon_n} - \tau_{L_{\hat{\tau}}^{\varepsilon_n}}^{\varepsilon_n} \geq \sum_{k=1}^{N_n - 1} (\tau_{\ell_k}^{\varepsilon_n} - \tau_{\ell_{k-1}}^{\varepsilon_n}) \stackrel{d}{=} \sum_{k=1}^{N_n - 1} H_k$$

with the last equality being in distribution conditionally on $\mathcal{F}_{\hat{\tau}}$, for H_k being conditionally independent and distributed as $H^{g(\ell_{k-1}) - \varepsilon_n \rightarrow g(\ell_k)}$. Clearly, ℓ_k is $\mathcal{F}_{\hat{\tau}}$ -measurable.

By the Laplace transform (7.8) of H_k and the Markov inequality, we get for $\lambda > 0$

$$\begin{aligned}
\mathbb{P}[J_{\hat{\tau}, \delta}^{\varepsilon_n} \geq N_n \mid \mathcal{F}_{\hat{\tau}}] &\leq \mathbb{P}\left[\sum_{k=1}^{N_n-1} H_k \leq \delta \mid \mathcal{F}_{\hat{\tau}}\right] \leq e^{\lambda\delta} \mathbb{E}\left[\exp\left(-\lambda \sum_{k=1}^{N_n-1} H_k\right) \mid \mathcal{F}_{\hat{\tau}}\right] \\
&= e^{\lambda\delta} \prod_{k=1}^{N_n-1} \mathbb{E}\left[\exp\left(-\lambda H^{g(\ell_{k-1})-\varepsilon_n \rightarrow g(\ell_k)}\right) \mid \mathcal{F}_{\hat{\tau}}\right] \\
&= e^{\lambda\delta} \prod_{k=1}^{N_n-1} \frac{\Phi_{\lambda,-}(g(\ell_{k-1}) - \varepsilon_n)}{\Phi_{\lambda,-}(g(\ell_k))} \leq e^{\lambda\delta} \prod_{k=1}^{N_n-1} \frac{\Phi_{\lambda,-}(g(\ell_k) - \varepsilon_n)}{\Phi_{\lambda,-}(g(\ell_k))} \\
&\leq e^{\lambda\delta} \left(\max_{0 \leq k < N_n} \frac{\Phi_{\lambda,-}(g(\ell_k) - \varepsilon_n)}{\Phi_{\lambda,-}(g(\ell_k))}\right)^{N_n-1} = e^{\lambda\delta} \left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1}
\end{aligned}$$

where $x_n := g(\ell_k)$ for the index $k = k_{n,\lambda}$ attaining the maximum.

2) For given $\delta > 0$ and $\hat{\tau} \leq T$, let us now consider the sequence $N_n = \lceil \varepsilon_0 / \varepsilon_n \rceil$, $n \in \mathbb{N}$. To investigate the limit $n \rightarrow \infty$, first observe that by Taylor expansion

$$\log \frac{\Phi_{\lambda,-}(x - \varepsilon_n)}{\Phi_{\lambda,-}(x)} = -\varepsilon_n \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)} + \varepsilon_n r(x, \varepsilon_n),$$

where $r(\cdot, \varepsilon_n) \rightarrow 0$ converges uniformly on compacts for $\varepsilon_n \rightarrow 0$. Since $\hat{\tau} + \delta \leq T + \delta$ is bounded, Lemma 7.6 yields a constant $M \in \mathbb{R}$ such that $\mathbb{P}[\exists n : x_n > M] \leq \frac{\eta}{2}$ for the x_n from above. On the event $\{\forall n : x_n \in I\}$ with compact $I := [g(0), M]$, we have uniform convergence of $r(x_n, \varepsilon_n)$ and thereby get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} e^{\lambda\delta} \left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1} &= \exp\left(\lambda\delta + \limsup_{n \rightarrow \infty} (N_n - 1) \log \frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right) \\
&= \exp\left(\lambda\delta + \limsup_{n \rightarrow \infty} (N_n \varepsilon_n - \varepsilon_n) \left(r(x_n, \varepsilon_n) - \frac{\Phi'_{\lambda,-}(x_n)}{\Phi_{\lambda,-}(x_n)}\right)\right) \\
&\leq \exp\left(\lambda\delta - \varepsilon_0 \inf_{x \in I} \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)}\right) = \sup_{x \in I} \exp\left(\lambda\delta - \varepsilon_0 \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)}\right).
\end{aligned}$$

By [PY03, Theorem 1], $\psi^x(\lambda) := \frac{1}{2} \Phi'_{\lambda,-}(x) / \Phi_{\lambda,-}(x)$ is the Laplace exponent of $A^x(\kappa^x)$, where κ^x is the inverse local time at constant level x of a (b, σ) -diffusion Z^x starting at x , and $A^x(t)$ is the occupation time

$$A^x(t) := \int_0^t \mathbb{1}_{\{Z_s^x \leq x\}} ds.$$

So we get $\exp(-2\varepsilon_0 \psi^x(\lambda)) = \mathbb{E}_x[\exp(-\lambda A^x(\kappa_{2\varepsilon_0}^x))] \rightarrow 0$ for $\lambda \rightarrow \infty$. By compactness of I and Dini's theorem there exists $\lambda = \lambda_{\varepsilon_0, \eta, M}$ such that for $\delta := 1/\lambda$ we have

$$\limsup_{n \rightarrow \infty} e^{\lambda\delta} \left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1} \leq e^{\lambda\delta} \sup_{x \in I} \exp(-2\varepsilon_0 \psi^x(\lambda)) \leq \frac{\eta}{2} \quad (7.14)$$

on the event $\{x_n \leq M \text{ for all } n\}$. Thanks to equation (7.13) and $\mathbb{P}[\exists n : x_n > M] \leq \eta/2$, this completes the proof. \square

Using the preceding two lemmas, we will first prove tightness of $(L^{\varepsilon_n})_n$ and of $(X^{\varepsilon_n})_n$ separately. Tightness of the pair $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ is handled afterwards.

Lemma 7.8 (Tightness of the local time approximations). *The sequence $(L^{\varepsilon_n})_n$ of càdlàg processes defined by (7.4) and (7.5) satisfies Aldous' criterion and thus is tight.*

Proof. Part (a) of Proposition 7.5 is clear, as the initial value $L_0^{\varepsilon_n} = \varepsilon_n$ is deterministic and $J_T(L^{\varepsilon_n}) \leq \varepsilon_n$. For part (b), consider $T, \eta, \varepsilon_0 > 0$ and any bounded L^{ε_n} -stopping time $\hat{\tau} \leq T$. The event $|L_{\hat{\tau}+\delta}^{\varepsilon_n} - L_{\hat{\tau}}^{\varepsilon_n}| \geq \varepsilon_0$ means that L^{ε_n} performs at least $N_n := \lceil \varepsilon_0/\varepsilon_n \rceil$ jumps in the stochastic interval $[\hat{\tau}, \hat{\tau} + \delta]$. Now, Lemma 7.7 yields some n_0 and $\delta_0 = \delta_0(\varepsilon_0)$ such that Aldous' criterion is satisfied for all $n \geq n_0$. Hence, $(L^{\varepsilon_n})_n$ is tight by Proposition 7.5. \square

We next obtain boundedness of $(X^{\varepsilon_n})_n$ that we will need for the proof of Lemma 7.10 to conclude tightness.

Lemma 7.9 (Bounding the diffusion approximations). *Let $T \in (0, \infty)$ and $\eta > 0$. Then there exists $M \in \mathbb{R}$ such that $\mathbb{P}[\sup_{t \in [0, T]} |X_t^{\varepsilon_n}| > M] < \eta$ for all $n \in \mathbb{N}$.*

Proof. By Lemma 7.6, for every $n \in \mathbb{N}$ the process X^{ε_n} on $[0, T]$ is bounded from above by a constant M with probability at least $1 - \eta/2$. It remains to show that it is also bounded from below with high probability. To this end, we will construct a process Y that is a lower bound for all X^{ε_n} and then argue for Y .

Let $\hat{\varepsilon} := \sup_n \varepsilon_n$ and consider a discretely reflected (b, σ) -diffusion Y starting in $Y_0 = y := g(0) - 2\hat{\varepsilon}$ with reflection at constant boundary $c := g(0) - \hat{\varepsilon}$ by jumps of size $-\hat{\varepsilon}$. So Y is a special case of our construction (7.4)–(7.5), for a constant boundary function: $dY_t = b(Y_t)dt + \sigma(Y_t)dW_t - L_t^Y$ with $L_t^Y := \sum_{0 \leq s \leq t} \Delta L_s^Y$ and $\Delta L_t^Y := \hat{\varepsilon} \mathbb{1}_{\{Y_{t-} = c\}}$. Let $\tau_k^Y := \inf\{t > 0 \mid L_t^Y > k\hat{\varepsilon}\}$ be the k -th hitting time of Y at the boundary c . So on all $[\tau_k^Y, \tau_{k+1}^Y]$, Y is a continuous (b, σ) -diffusion starting in y . Now for fixed $n, \varepsilon := \varepsilon_n$, note that $X_{\tau_{m\varepsilon}^{\varepsilon}}^{\varepsilon} = g((m-1)\varepsilon) - \varepsilon \geq c \geq Y_{\tau_{m\varepsilon}^{\varepsilon}}^{\varepsilon}$ by monotonicity of g . As $\tau_{m\varepsilon}^{\varepsilon} \rightarrow \infty$ for $m \rightarrow \infty$ by Lemma 7.2, induction over the inverse (discrete) local times $\tau_{m\varepsilon}^{\varepsilon}$, $m \in \mathbb{N}$, yields $X^{\varepsilon} \geq Y$ on $[\tau_k^Y, \tau_{k+1}^Y]$ if $X_{\tau_k^Y}^{\varepsilon} \geq Y_{\tau_k^Y}^{\varepsilon}$ by comparison results [KS91, Theorem 5.2.18]. Since $X_0^{\varepsilon} \geq Y_0$, the latter follows by induction over k . As $\tau_k^Y \rightarrow \infty$ for $k \rightarrow \infty$ by Lemma 7.2, one has $X^{\varepsilon_n} \geq Y$ on $[0, \infty)$ for all n . So it suffices to show $\mathbb{P}[\inf_{t \in [0, T]} Y_t < -M] < \eta/2$ for some M .

Using (7.13), we can bound the number of jumps of Y in $[0, T]$ by some constant N with high probability (see (7.13) and note that part 1) of the proof for Lemma 7.7 was for an arbitrary N_n), to obtain $\mathbb{P}[\tau_N^Y \leq T] \leq \eta/4$ for some $N \in \mathbb{N}$. An “excursion” of Y towards $-M$ will happen during one of the intervals $[\tau_k^Y, \tau_{k+1}^Y]$, so

$$\mathbb{P}\left[\inf_{t \in [0, T]} Y_t < -M, \tau_N^Y > T\right] \leq N\mathbb{P}[H^{y \rightarrow -M} \leq T] < \eta/4$$

for $M = M_N$ large enough. \square

Lemma 7.10 (Tightness of the reflected diffusion approximations). *The sequence $(X^{\varepsilon_n})_n$ of càdlàg processes from (7.4) and (7.5) satisfies Aldous' criterion and thus is tight.*

Proof. Condition (a) of Proposition 7.5 is clearly met. To verify Aldous's criterion 7.5(b), let $\eta > 0$, $T \in (0, \infty)$, and $\hat{\tau} \leq T$ be a stopping time. Let us consider the events $\{X_{\hat{\tau}+\delta}^{\varepsilon_n} \leq X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_0\} \cup \{X_{\hat{\tau}+\delta}^{\varepsilon_n} \geq X_{\hat{\tau}}^{\varepsilon_n} + \varepsilon_0\} = \{|X_{\hat{\tau}+\delta}^{\varepsilon_n} - X_{\hat{\tau}}^{\varepsilon_n}| \geq \varepsilon_0\}$ separately. By Lemma 7.9, $|X_{\hat{\tau}}^{\varepsilon_n}|$ is with a probability of at least $1 - \eta/4$ bounded by some constant M (not depending on n and $\hat{\tau}$).

1) First consider $\{X_{\hat{\tau}+\delta}^{\varepsilon_n} \leq X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_0\}$. For $\xi := X_{\hat{\tau}}^{\varepsilon_n}$ we construct a reflected process Y^ξ such that $Y_t^\xi \leq X_{\hat{\tau}+t}^{\varepsilon_n}$ for all $t \geq 0$. To this end, choose $\hat{\varepsilon} \leq \varepsilon_0/4$ and n large enough such that $\varepsilon_n \leq \hat{\varepsilon}$, and let $(Y_t^\xi)_{t \geq 0}$ be the (b, σ) -diffusion w.r.t. the Brownian motion $(W_{\hat{\tau}+t} - W_{\hat{\tau}})_{t \geq 0}$ which starts at $Y_0^\xi = \xi - 2\hat{\varepsilon}$ and is (discretely) reflected by jumps of size $-\hat{\varepsilon}$ at the constant boundary at level $\xi - \hat{\varepsilon}$. More precisely, $dY_t^\xi = b(Y_t^\xi) dt + \sigma(Y_t^\xi) dW_{\hat{\tau}+t} - K_t^\xi$ with (discrete) local time $K_t^\xi := \sum_{0 \leq s \leq t} \Delta K_s^\xi$ for $\Delta K_t^\xi := \hat{\varepsilon} \mathbb{1}_{\{Y_{t-}^\xi = \xi - \hat{\varepsilon}\}}$. Global existence and uniqueness of (Y^ξ, K^ξ) follows from proof of Lemma 7.2. Using comparison arguments and induction as in the proof of Lemma 7.9, one verifies that $Y_t^\xi \leq X_{\hat{\tau}+t}^{\varepsilon_n}$ for $t \in [0, \infty)$. Noting that $Y_\delta^\xi \leq X_{\hat{\tau}+\delta}^{\varepsilon_n}$ and using the strong Markov property of Y^ξ w.r.t. the filtration $(\mathcal{F}_{\hat{\tau}+t})_{t \geq 0}$, we get

$$\mathbb{P}[X_{\hat{\tau}+\delta}^{\varepsilon_n} \leq X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_0, |X_{\hat{\tau}}^{\varepsilon_n}| \leq M] \leq \sup_{-M \leq x \leq M} \mathbb{P}[Y_\delta^x \leq x - \varepsilon_0]. \quad (7.15)$$

By construction Y^ξ depends on n and τ (through ξ), while the right-hand side of (7.15) does not. Thus one only needs to bound the probability of an $(\varepsilon_0 - 2\hat{\varepsilon})$ -displacement of diffusions Y^x with starting points $x - 2\hat{\varepsilon}$ from a compact set, which are reflected (by $(-\hat{\varepsilon})$ -jumps) at constant boundaries $x - \hat{\varepsilon}$.

By the arguments in the proof of Lemma 7.7 (here applied for Y^x which is reflected at a constant boundary), for $\delta = \delta_0 > 0$ there exists $N \in \mathbb{N}$ with the following property: for every $x \in [-M, M]$, the number $J_\delta^x := \inf\{k \mid k\hat{\varepsilon} \geq K_\delta^x\}$ of jumps of Y^x until time δ is bounded by $N - 1$ with probability at least $1 - \eta/8$.

Indeed, by (7.13), fixing $\delta > 0$, $\lambda := 1/\delta$ one gets for any x that $\mathbb{P}[J_\delta^x \geq \lceil N(x) \rceil] \leq \eta/8$ where $N(x) := 1 + (\log(\eta/8) - 1)/(\log \Phi_{\lambda,-}(x - \hat{\varepsilon}) - \log \Phi_{\lambda,-}(x)) \in \mathbb{R}$. Compactness of $[-M, M]$ and continuity of $N(x)$ gives $N := \lceil \sup_{x \in [-M, M]} N(x) \rceil < \infty$. Hence,

$$\sup_{x \in [-M, M]} \mathbb{P}[Y_\delta^x \leq x - \varepsilon_0, J_\delta^x \leq N - 1] \leq N \sup_{x \in [-M, M]} \mathbb{P}[H^{x-2\hat{\varepsilon}} \rightarrow x - \varepsilon_0 \leq \delta],$$

since for the event under consideration, the process Y^x would have to move at least once (in at most N occasions) continuously from $x - 2\hat{\varepsilon}$ to $x - \varepsilon_0$. Choosing $d := (\varepsilon_0 - 2\hat{\varepsilon})/2 > 0$, $K := \lfloor 2M/d \rfloor$ and $y_k := kd - M$, we get

$$\begin{aligned} \mathbb{P}[H^{X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_n} \rightarrow X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_0 \leq \delta, |X_{\hat{\tau}}^{\varepsilon_n}| \leq M] &\leq \eta/8 + N \sup_{x \in [-M, M]} \mathbb{P}[H^{x-2\hat{\varepsilon}} \rightarrow x - \varepsilon_0 \leq \delta] \\ &= \eta/8 + N \max_{k=0, \dots, K} \sup_{x \in [kd-M, (k+1)d-M]} \mathbb{P}[H^{x-2\hat{\varepsilon}} \rightarrow x - \varepsilon_0 \leq \delta] \\ &\leq \eta/8 + N \max_{k=-2, \dots, K} \mathbb{P}[H^{y_k} \rightarrow y_k - d \leq \delta]. \end{aligned} \quad (7.16)$$

For a sufficiently small $\delta = \delta_1 \in (0, \delta_0]$ the right-hand side of (7.16) can be made smaller than $\eta/4$. The above holds for all n such that $\varepsilon_n \leq \hat{\varepsilon}$, meaning that there is

some n_0 such that it holds for all $n \geq n_0$. Note that δ_1 only depends on T (via M and K) and on n_0 but not on n . Hence, for all $\delta \in (0, \delta_1]$, all $n \geq n_0$ and all $\hat{\tau} \leq T$ we have

$$\mathbb{P}[X_{\hat{\tau}+\delta}^{\varepsilon_n} \leq X_{\hat{\tau}}^{\varepsilon_n} - \varepsilon_0] \leq \frac{\eta}{2}. \quad (7.17)$$

2) For the alternative second case $X_{\hat{\tau}+\delta}^{\varepsilon_n} \geq X_{\hat{\tau}}^{\varepsilon_n} + \varepsilon_0$, consider the solution $(Y_t)_{t \geq \hat{\tau}}$ on $[\hat{\tau}, \infty]$ of $dY_t = b(Y_t) dt + \sigma(Y_t) dW_t$ with $Y_{\hat{\tau}} = X_{\hat{\tau}}^{\varepsilon_n}$. Using comparison results for continuous diffusions [KS91, Theorem 5.2.18] inductively over times $[\tau_{(k-1)\varepsilon_n}^{\varepsilon_n}, \tau_{k\varepsilon_n}^{\varepsilon_n}]$, we find $Y_t \geq X_t^{\varepsilon_n}$ for all $t \in [\hat{\tau}, \infty]$, a.s. Hence, arguing like in the previous case

$$\begin{aligned} \mathbb{P}[X_{\hat{\tau}+\delta}^{\varepsilon_n} \geq X_{\hat{\tau}}^{\varepsilon_n} + \varepsilon_0, |X_{\hat{\tau}}^{\varepsilon_n}| \leq M] &\leq \mathbb{P}[Y_{\hat{\tau}+\delta} \geq Y_{\hat{\tau}} + \varepsilon_0, |Y_{\hat{\tau}}| \leq M] \\ &\leq \sup_{-M \leq y \leq M} \mathbb{P}[H^{y \rightarrow y+\varepsilon_0} \leq \delta]. \end{aligned} \quad (7.18)$$

As in (7.16) we find some $\delta_2 > 0$ such that for all $\delta \in (0, \delta_2]$ the right-hand side of (7.18) is bounded by $\eta/4$. Hence we have $\mathbb{P}[X_{\hat{\tau}+\delta}^{\varepsilon_n} \geq X_{\hat{\tau}}^{\varepsilon_n} + \varepsilon_0] \leq \eta/2$, so with (7.17), Proposition 7.5 applies. \square

Having two tight sequences of càdlàg processes, it is not clear in general whether the pair is also tight in the product space, cf. [SK85]. But we can utilize that fact that $(X^{\varepsilon_n})_n$ and $(L^{\varepsilon_n})_n$ satisfy Aldous' criterion and that their jump times and jump magnitudes are identical.

Lemma 7.11 (Tightness of joint approximations). *The sequence $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ of càdlàg \mathbb{R}^2 -valued processes defined by (7.4) and (7.5) is tight.*

Proof. In view of Proposition 7.5, choose the space $E := \mathbb{R}^2$ equipped with Euclidean norm $|\cdot|$ and let $Y^n := (X^{\varepsilon_n}, L^{\varepsilon_n}) \in D([0, \infty), E)$. Then $Y_0^n = (-\varepsilon_n, \varepsilon_n)$ and $J_T(Y^n) = \sqrt{2}\varepsilon_n$ form tight sequences in E and \mathbb{R} , respectively. Furthermore,

$$\mathbb{P}[|Y_{\hat{\tau}+\delta}^n - Y_{\hat{\tau}}^n| \geq \varepsilon_0] \leq \mathbb{P}\left[|X_{\hat{\tau}+\delta}^{\varepsilon_n} - X_{\hat{\tau}}^{\varepsilon_n}| \geq \frac{\varepsilon_0}{2}\right] + \mathbb{P}\left[|L_{\hat{\tau}+\delta}^{\varepsilon_n} - L_{\hat{\tau}}^{\varepsilon_n}| \geq \frac{\varepsilon_0}{2}\right].$$

Hence Y^n also satisfies Aldous's criterion and therefore is tight. \square

Tightness only implies weak convergence of a subsequence. It remains to show (in Lemma 7.13) that every limit point satisfies (7.1) and (7.2) and that uniqueness in law holds. The latter will follow from pathwise uniqueness results for SDEs with reflection, while for the former we apply results due to [KP96] on weak converges of SDEs. For that purpose, note that the approximated local times form a *good* sequence of semimartingales (cf. [KP96, Definition 7.3]), as shown in the following

Lemma 7.12. *The sequence $(L^{\varepsilon_n})_n$ is of uniformly controlled variation and thus good.*

Proof. Let $\delta := \sup_n \varepsilon_n$. Then all processes L^{ε_n} have jumps of size at most $\delta < \infty$. Fix some $\alpha > 0$. By tightness, there exists some $C \in \mathbb{R}$ such that $\mathbb{P}[L_{\alpha}^{\varepsilon_n} > C] \leq 1/\alpha$. So the

stopping time $\tau_{n,\alpha} := \inf\{t \geq 0 \mid L_t^{\varepsilon_n} > C\}$ satisfies $\mathbb{P}[\tau_{n,\alpha} \leq \alpha] = \mathbb{P}[L_\alpha^{\varepsilon_n} > C] \leq 1/\alpha$. Moreover, by monotonicity of L^{ε_n} we have

$$\mathbb{E}\left[\int_0^{t \wedge \tau_{n,\alpha}} d|L^{\varepsilon_n}|_s\right] = \mathbb{E}[L_{t \wedge \tau_{n,\alpha}}^{\varepsilon_n}] \leq C < \infty.$$

Hence (L^{ε_n}) is of uniformly controlled variation in the sense of [KP96, Definition 7.5]. So by [KP96, Theorem 7.10] it is a *good* sequence of semimartingales. \square

We have gathered all necessary results to prove convergence of our approximating diffusion and local time to the continuous counterpart.

Lemma 7.13 (Weak convergence of the approximations). *The sequence $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ of càdlàg processes defined by (7.4) – (7.5) converges weakly to the unique continuous strong solution (X, L) of (7.1) – (7.2).*

Proof. By Prokhorov's theorem, tightness of $(X^{\varepsilon_n}, L^{\varepsilon_n}, W)_n$ implies weak convergence of a subsequence to some limit point, $(X^{\varepsilon_{n_k}}, L^{\varepsilon_{n_k}}, W)_k \Rightarrow (\tilde{X}, \tilde{L}, \tilde{W}) \in D([0, \infty), \mathbb{R}^3)$. Continuity of (\tilde{X}, \tilde{L}) is clear since $\varepsilon_n \rightarrow 0$ is the maximum jump size. First we prove that (\tilde{X}, \tilde{L}) satisfies the asserted SDEs. Afterwards, we will prove uniqueness of the limit point. To ease notation, let w.l.o.g. the subsequence (n_k) be (n) .

By [KP96, Theorem 8.1] we get that (\tilde{X}, \tilde{L}) satisfy (7.1) for the semimartingale \tilde{W} . That \tilde{W} is a Brownian motion follows from standard arguments, cf. [NÖ10, proof of Thm. 1.9]. As $D([0, \infty), \mathbb{R}^3)$ is separable we find, by an application of the Skorokhod representation theorem, that \tilde{L} is non-decreasing and $\tilde{X}_t \leq g(\tilde{L}_t)$ for all $t \geq 0$, \mathbb{P} -a.s. because these properties already hold for $(X^{\varepsilon_n}, L^{\varepsilon_n})$.

To prove that \tilde{L} grows only at times t with $\tilde{X}_t = g(\tilde{L}_t)$, we have to approximate the indicator function by continuous functions. For $\delta > 0$ define

$$h_\delta(x, \ell) := \begin{cases} (x - g(\ell))/\delta + 1 & \text{for } g(\ell) - \delta \leq x \leq g(\ell), \\ 1 - (x - g(\ell))/\delta & \text{for } g(\ell) \leq x \leq g(\ell) + \delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$h_0(x, \ell) := \mathbb{1}_{\{x=g(\ell)\}} \text{ and } H_t^{\delta,n} := h_\delta(X_t^{\varepsilon_n}, L_t^{\varepsilon_n}) \text{ and } \tilde{H}_t^\delta := h_\delta(\tilde{X}_t, \tilde{L}_t).$$

For $\delta \searrow 0$ the functions $h_\delta \searrow h_0$ converge pointwise monotonically. Continuity of h_δ implies weak convergence $(H^{\delta,n}, L^{\varepsilon_n}) \Rightarrow (\tilde{H}^\delta, \tilde{L})$. By Lemma 7.12, (L^{ε_n}) is a good sequence; so for every $\delta > 0$ the stochastic integrals $\int_0^\cdot H_{s-}^{\delta,n} dL_s^{\varepsilon_n} \Rightarrow \int_0^\cdot \tilde{H}_{s-}^\delta d\tilde{L}_s$ converge weakly. Note that $dL_t^{\varepsilon_n} = H_{t-}^{0,n} dL_t^{\varepsilon_n}$. Hence, for every $\delta > 0$ we have

$$\int_0^\cdot H_{s-}^{\delta,n} dL_s^{\varepsilon_n} = \int_0^\cdot H_{s-}^{\delta,n} H_{s-}^{0,n} dL_s^{\varepsilon_n} = \int_0^\cdot H_{s-}^{0,n} dL_s^{\varepsilon_n} = L^{\varepsilon_n}.$$

With weak convergence $L^{\varepsilon_n} \Rightarrow \tilde{L}$ it follows for every $\delta > 0$ that $\tilde{L}_t = \int_0^t \tilde{H}_{s-}^\delta d\tilde{L}_s$. By monotonicity of \tilde{L} , $d\tilde{L}_t$ defines a random measure on $[0, \infty)$. Hence monotone convergence of $\tilde{H}_t^\delta \searrow \tilde{H}_t^0$ yields $d\tilde{L}_t = h_0(\tilde{X}_t, \tilde{L}_t) d\tilde{L}_t$.

Thus, we showed that $(X^\varepsilon, L^\varepsilon)$ converges in distribution to a weak solution (\tilde{X}, \tilde{L}) of the reflected SDE, i.e. it might be defined on a different probability space with its own Brownian motion. Note that (\tilde{X}, \tilde{L}) is continuous on $[0, \infty)$ and that $\tilde{\tau}_\infty := \sup_k \tilde{\tau}_k = \infty$ a.s., where $\tilde{\tau}_k := \inf\{t > 0 \mid |\tilde{X}_t| \vee \tilde{L}_t > k\}$. To show the existence and uniqueness of a strong solution as stated in the theorem, we will use the results from [DI93]. Consider the domain $\bar{G} := \{(x, \ell) \in \mathbb{R}^2 \mid x \leq g(\ell), \ell \geq 0\}$. We may interpret the process (X_t, L_t) as a continuous diffusion in \bar{G} with oblique reflection in direction $(-1, +1)$ at the boundary, although the notion of a two-dimensional reflection seems unusual here, because (X, L) only varies in one dimension in the interior of G . The unbounded domain G can be exhausted by bounded domains $G_k := \{(x, \ell) \in G \mid |x|, |\ell| < k\}$, which might have a non-smooth boundary especially at $(g(0), 0)$, but still satisfy [DI93, Cond. (3.2)]. Hence, by [DI93, Cor. 5.2] the process (X, L) exists (up to explosion time) on the initial probability space and is (strongly) unique on $\llbracket 0, \tau_k \rrbracket$ with exit time $\tau_k := \inf\{t > 0 \mid |X_t| \vee L_t > k\}$, for all $k \in \mathbb{N}$. So (X, L) is unique until explosion time $\tau_\infty := \sup_k \tau_k$. Moreover, by [DI93, Theorem 5.1] we have the following pathwise uniqueness result: for any two continuous solutions (X^1, L^1) and (X^2, L^2) with explosion times τ_∞^1 and τ_∞^2 , respectively defined on the same probability space with the same Brownian motion and the same initial condition, we have that $X^1 = X^2$ and $L^1 = L^2$ on $\llbracket 0, \tau_k^1 \wedge \tau_k^2 \rrbracket$ for every $k \in \mathbb{N}$ a.s. Using a known argument due to Yamada and Watanabe, ideas being as in [KS91, Ch.5.3D], one can bring the two (weak) solutions $(\tilde{X}, \tilde{L}, \tilde{W})$ and (X, L, W) to a canonical space with a common Brownian motion. By pathwise uniqueness there, one concludes that $\tau_\infty = \infty$ a.s. (as $\tilde{\tau}_\infty = \infty$); hence the strong solution (X, L) does not explode in finite time. In addition, we conclude uniqueness in law like in [KS91, Prop.5.3.20] and thus any weak limit point of the approximating sequence $(X^\varepsilon, L^\varepsilon)$ will have the same law as (X, L) . \square

Corollary 7.14 (Weak convergence of the inverse local times). *For every $\ell > 0$, the sequence $(\tau_\ell^{\varepsilon_n})_n$ defined by (7.6) converges in distribution to the inverse local time τ_ℓ defined by (7.3).*

Proof. Convergence $L^{\varepsilon_n} \Rightarrow L$ implies $L_t^{\varepsilon_n} \Rightarrow L_t$ at all continuity points of L , i.e. at all points, hence $\mathbb{P}[\tau_\ell^{\varepsilon_n} \leq t] = \mathbb{P}[L_t^{\varepsilon_n} \geq \ell] \rightarrow \mathbb{P}[L_t \geq \ell] = \mathbb{P}[\tau_\ell \leq t]$. \square

This completes the proof of Theorem 7.3.

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